# DRINFELD ASSOCIATORS, BRAID GROUPS AND EXPLICIT SOLUTIONS OF THE KASHIWARA–VERGNE EQUATIONS

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Abstract. The Kashiwara–Vergne (KV) conjecture states the existence of solutions of a pair of equations related with the Campbell–Baker–Hausdorff series. It was solved by Meinrenken and the first author over $\mathbb{R}$, and in a formal version, by the first and last authors over a field of characteristic 0. In this paper, we give a simple and explicit formula for a map from the set of Drinfeld associators to the set of solutions of the formal KV equations. Both sets are torsors under the actions of prounipotent groups, and we show that this map is a morphism of torsors. When specialized to the KZ associator, our construction yields a solution over $\mathbb{R}$ of the original KV conjecture.

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## Introduction

In [KV], M. Kashiwara and M. Vergne formulated a conjecture on the form of the Campbell–Baker–Hausdorff (CBH) series. This conjecture triggered the work of several authors (for a review see [T2]). In particular, Kashiwara–Vergne settled it for solvable Lie algebras ([KV]), Rouvière gave a proof for $\mathfrak{sl}_2$ ([R]), and Vergne ([V]) and Alekseev–Meinrenken ([AM1]) proved it for quadratic Lie algebras. All these constructions lead to explicit rational formulas for solutions of the KV conjecture. The general case was settled in the positive by Alekseev–Meinrenken ([AM2]) using Kontsevich's deformation quantization theory and results in [T1]. The corresponding solution is defined over $\mathbb{R}$, and expresses as an infinite series where coefficients are combinations of Kontsevich integrals on configuration spaces and integrals over simplices. The values of most of these coefficients remain unknown.

Later, the first and last authors gave another proof ([AT]), based on Drinfeld's theory of associators. In that paper, the Kashiwara–Vergne (KV) conjecture was reformulated as the problem of constructing special automorphisms of the free Lie algebra with two generators with coboundary Jacobian (see Section 2); the authors also showed that each associator gives rise to an affine line of such automorphisms. The solution is defined as a nonabelian cochain with coboundary equal to the associator. Such a construction is inspired by the theory of quantization of Lie bialgebras, and the existence problem is solved by showing that obstructions vanish in all degrees.

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The purpose of the present work is to give a direct construction of the map $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$, $\Phi \mapsto \mu_\Phi$ from associators to solutions of the KV equations (we work over a field $\mathbf{k}$ of characteristic 0). Namely, for $\Phi \in M_1(\mathbf{k})$, $\mu_\Phi$ is the automorphism of the topologically free Lie algebra generated by $x, y$ given by

(1)
$$\mu_\Phi : x \mapsto \Phi(x,-x-y)x\Phi(x,-x-y)^{-1}, \quad y \mapsto e^{-(x+y)/2}\Phi(y,-x-y)y\Phi(y,-x-y)^{-1}e^{(x+y)/2}.$$

Our main result (Theorem 2.1) is the identity

$$\Phi(t_{12},t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}. \tag{2}$$

This identity implies that the Jacobian of $\mu_\Phi$ is a cocycle, and therefore a coboundary according to cohomology computations in [AT]; it can then be expressed using the $\Gamma$-function $\Gamma_\Phi$ of $\Phi$ (see [DT, E]). Identity (2) also implies that $\mu_\Phi$ is special, i.e., satisfies

$$\mu_\Phi(\log(e^x e^y)) = x + y \tag{3}$$

(see Subsection 5.2 and also Proposition 7.4 in [AT]); we also give a direct proof of (3) based on the hexagon and duality identities satisfied by $\Phi$. The conjunction of (3) and of the fact that the Jacobian of $\mu_\Phi$ is a coboundary actually means that $\mu_\Phi$ is a solution of the KV equations introduced in [AT].

The affine line of solutions of the KV equations attached in [AT] to $\Phi$ then takes the form $\{\mathrm{Inn}(e^{s(x+y)})\circ\mu_\Phi, s\in\mathbf{k}\}$, where $\mathrm{Inn}(g)=(u\mapsto gug^{-1})$. It remains an open question whether all the solutions of the KV equation are of this form.

The strategy for proving (2) is as follows. For each associator $\Phi$ and each parenthesization $O$ of a word in $n$ identical letters (the letter is $\bullet$), Drinfeld and Bar-Natan define an isomorphism $\tilde\mu_\Phi^O : \mathrm{PB}_n(\mathbf{k})\to\exp(\hat{\mathfrak{t}}_n)$ from the prounipotent completion of the pure braid group with $n$ strands to the group associated with the holonomy Lie algebra. Note that $\mathrm{PB}_n$ contains the free group $\mathrm{F}_{n-1}$ as a normal subgroup, while $\mathfrak{t}_n$ contains the free Lie algebra $\mathfrak{f}_{n-1}$ as an ideal; we show that the above isomorphisms restrict to isomorphisms $\mu_\Phi^O : \mathrm{F}_{n-1}(\mathbf{k})\to\exp(\hat{\mathfrak{f}}_{n-1})$ (in the case of the left parenthesization, this was proved in [HM]). We note that $\mu_\Phi$ may be interpreted as the isomorphism $\mathrm{F}_2(\mathbf{k})\to\exp(\hat{\mathfrak{f}}_2)$ corresponding to $\bullet(\bullet\bullet)$, so $\mu_\Phi=\mu_{\bullet(\bullet\bullet)}$ (we write $\mu_O$ instead of $\mu_\Phi^O$ when no confusion is possible). We then show the identity

$$\mu_{O^{(i)}}=\mu_O^{1,2,\ldots,ii+1,\ldots,n}\circ\mu_{\bullet(\bullet\bullet)}^{i,i+1}, \tag{4}$$

where $O$ is a parenthesized word of length $n$ and $O^{(i)}$ is the parenthesized word obtained from it by replacing the $(i+1)$th letter $\bullet$ by $(\bullet\bullet)$. Applying this identity to $O=\bullet(\bullet\bullet)$ with $i=1,2$ and using the identity $\mu_\Phi^{O'}=\mathrm{Ad}(\Phi_{O,O'})\circ\mu_\Phi^O$ relating the various $\mu_\Phi^O$, we obtain (2).

We then study the torsor aspects of the map $\Phi\mapsto\mu_\Phi$. While $M_1(\mathbf{k})$ is a torsor under the commuting actions of the groups $\mathrm{GT}_1(\mathbf{k})$ and $\mathrm{GRT}_1(\mathbf{k})$, $\mathrm{SolKV}(\mathbf{k})$ is a torsor under the actions of groups $\mathrm{KV}(\mathbf{k})$ and $\mathrm{KRV}(\mathbf{k})$. We prove that $\Phi\mapsto\mu_\Phi$ is a morphism of torsors, i.e., there exist group morphisms $\mathrm{GT}_1(\mathbf{k})\to\mathrm{KV}(\mathbf{k})$, $f\mapsto\alpha_f$ and $\mathrm{GRT}_1(\mathbf{k})\to\mathrm{KRV}(\mathbf{k})$, $g\mapsto a_g$, compatible with the actions (the Lie algebra version of the latter morphism was already constructed in [AT]). We give a direct proof of these facts, based on the nonemptiness of $M_1(\mathbf{k})$ (a result in [Dr]); we also sketch an independent proof of $\alpha_f\in\mathrm{KV}(\mathbf{k})$; its main ingredient is the identity

$$\mathrm{Ad}\,f(x_{12},x_{23})\circ\alpha_f^{\widetilde{12},3}\circ\alpha_f^{1,2}=\alpha_f^{1,\widetilde{23}}\circ\alpha_f^{2,3}. \tag{5}$$

A similar independent proof of $a_g\in\mathrm{KRV}(\mathbf{k})$ may be given based on

$$\mathrm{Ad}\,g(t_{12},t_{23})\circ a_g^{12,3}\circ a_g^{1,2}=a_g^{1,23}\circ a_g^{2,3}.$$

It can be proved using the techniques of [AT] that the sets of solutions of both equations are affine lines, and our result gives explicit formulas for these solutions. We also observe that (5) can be generalized to the profinite and pro-$l$ setups (i.e., we have morphisms $\widehat{\mathrm{GT}}\to\mathrm{Aut}(\widehat{\mathrm{F}}_2)$ and $\mathrm{GT}_l\to\mathrm{Aut}((\mathrm{F}_2)_l)$, $f\mapsto\alpha_f$, and (5) takes place in $\mathrm{Aut}(\widehat{\mathrm{F}}_3)$ or $\mathrm{Aut}((\mathrm{F}_3)_l)$).

Formula (4) and its analogue (5) then enable us to compute the Jacobians of $\mu_\Phi^O$ : $\mathrm{F}_{n-1}(\mathbf{k})\to\exp(\mathfrak{f}_{n-1})$ and $\alpha_f^O\in\mathrm{Aut}(\mathrm{F}_{n-1}(\mathbf{k}))$, where $O$ is an arbitrary parenthesized word, $\Phi\in M_1(\mathbf{k})$, $f\in\mathrm{GT}_1(\mathbf{k})$, in terms of in terms of $\Gamma_\Phi$ and of the '$\Gamma$-function' of $f$.

Finally, we show that specializing our construction to the Knizhnik–Zamolodchikov (KZ) associator yields an explicit solution of the original KV conjecture, where the Lie series are required to converge for any finite dimensional Lie algebra and the Duflo series is required to coincide with the generating series of Bernoulli numbers.

**Acknowledgements.** We are grateful to V.G. Drinfeld and to D. Bar-Natan who posed the question of how to construct explicit solutions of the KV problem in terms of associators. The formula for $\mu_\Phi$ as well as the proof of equation (3) were suggested to us by D. Calaque. We would like to thank G. Massuyeau for discussions and for pointing out reference [HM].

The research of A.A. was supported in part by the grants 200020-120042 and 200020-121675 of the Swiss National Science Foundation, and the research of C.T. was supported by CNRS.

Contents

## 1. Preliminary results

In this section, we recall the notions of tangential derivations and automorphisms of free Lie algebras, their divergence and Jacobian cocycles, the actions of pure braid groups (resp., infinitesimal braid Lie algebras) on free groups Lie algebras by tangential automorphisms (resp., derivations), and simplicial morphisms between these objects.

1.1. **Tangential automorphisms, the Jacobian cocycle, and complexes.** Let $\mathfrak{f}_n$ be the free Lie algebra with generators $x_1,...,x_n$, $\hat{\mathfrak{f}}_n$ its degree completion (where the generators $x_k$ have degree 1). For $u_1,...,u_n \in \mathfrak{f}_n$, we denote by $[\![u_1,...,u_n]\!]$ the derivation of $\mathfrak{f}_n$ given by $x_k \mapsto [u_k,x_k]$. In this way, we define a linear map $(\mathfrak{f}_n)^n \to \mathrm{Der}(\mathfrak{f}_n)$. Its image is a (positively) graded Lie subalgebra $\mathfrak{tder}_n$ of $\mathrm{Der}(\mathfrak{f}_n)$; its elements are called the tangential derivations of $\mathfrak{f}_n$. We similarly define $\mathfrak{tder}_n^\wedge \subset \mathrm{Der}(\hat{\mathfrak{f}}_n)$ as the degree completion of $\mathfrak{tder}_n$; it is a pronilpotent Lie algebra.

If $U_1,...,U_n \in \exp(\hat{\mathfrak{f}}_n)$, we similarly define $[\![U_1,...,U_n]\!]$ as the automorphism of $\hat{\mathfrak{f}}_n$ given by $x_k \mapsto U_k x_k U_k^{-1}$. This defines a map $\exp(\hat{\mathfrak{f}}_n)^n \to \mathrm{Aut}(\hat{\mathfrak{f}}_n)$, whose image is the subgroup of tangential automorphisms $\mathrm{TAut}_n \subset \mathrm{Aut}(\hat{\mathfrak{f}}_n)$. The exponential sets up an isomorphism $\exp : \mathfrak{tder}_n^\wedge \to \mathrm{TAut}_n$.

Define $\mathfrak{T}_n := A_n/[A_n,A_n]$ as the quotient of the free associative algebra $A_n \simeq U(\mathfrak{f}_n)$ by its subspace of commutators; this is the vector space spanned by the set of cyclic words in $x_1,...,x_n$. $\mathfrak{T}_n$ is equipped with an action of $\mathrm{Der}(\mathfrak{f}_n)$, induced by the action of $\mathrm{Der}(\mathfrak{f}_n)$ on $A_n$. We denote by $x \mapsto \langle x\rangle$ the canonical projection map $A_n \to \mathfrak{T}_n$. $\mathfrak{T}_n$ is positively graded and we denote by $\hat{\mathfrak{T}}_n$ its degree completion; it is equipped with actions of $\mathrm{Der}(\hat{\mathfrak{f}}_n)$ and $\mathrm{Aut}(\hat{\mathfrak{f}}_n)$.

One shows that any $u \in \mathfrak{tder}_n$ can be written as $u = [\![u_1,...,u_n]\!]$, where $(u_1,...,u_n)$ is uniquely determined by the condition $p_1(u_1) = ... = p_n(u_n) = 0$, where $p_k : \mathfrak{f}_n \to \mathbf{k}$ is the linear map such that $u = \sum_k p_k(u)x_k$ modulo $[\mathfrak{f}_n,\mathfrak{f}_n]$.

We define simplicial group morphisms $\mathrm{TAut}_n \to \mathrm{TAut}_m$ as follows. Let[1] $\phi : [m] \supset D_\phi \to [n]$ be a partially defined map, and let $(a_1,...,a_n) \in (\mathfrak{f}_n)^n$ be such that each $a_k$ has vanishing linear term in $x_k$. We set $[\![a_1,...,a_n]\!]^\phi := [\![b_1,...,b_m]\!]$, where $b_\ell(x_1,...,x_m) := a_{\phi(\ell)}(\sum_{k\in\phi^{-1}(1)} x_k, ..., \sum_{k\in\phi^{-1}(n)} x_k)$. This formula defines a Lie algebra morphism $\mathfrak{tder}_n \to \mathfrak{tder}_m$, which induces a group morphism $\mathrm{TAut}_n \to \mathrm{TAut}_m$, also denoted $x \mapsto x^\phi$. We will also use the notation $x^\phi = x^{\phi^{-1}(1),...,\phi^{-1}(n)}$. For example, $[\![a_1,a_2]\!]^{12,3} = [\![a_1(x_1+x_2,x_3), a_1(x_1+x_2,x_3), a_2(x_1+x_2,x_3)]\!]$.

We also define noncommutative variants of these morphisms as follows. Let $\tilde\phi$ be a pair consisting of a partially defined map $\phi : [m] \supset D_\phi \to [n]$ as above and of total orders on each of the sets $\phi^{-1}(1),...,\phi^{-1}(n)$. We define $[\![a_1,...,a_n]\!]^{\tilde\phi} := [\![\tilde b_1,...,\tilde b_m]\!]$, where $\tilde b_\ell(x_1,...,x_m) := a_{\phi(\ell)}(\mathrm{cbh}(x_k|k\in\phi^{-1}(1)),...,\mathrm{cbh}(x_k|k\in\phi^{-1}(n)))$; here $\mathrm{cbh}(a_1,...,a_p) = \log(e^{a_1}...e^{a_p})$ and $\mathrm{cbh}(a_s|s\in S)$ is defined similarly, for $S$ a finite ordered set. We use the notation $x^{\tilde\phi} = x^{\widetilde{\phi^{-1}(1)},...,\widetilde{\phi^{-1}(n)}}$ (where the elements of $\phi^{-1}(k)$ are written in increasing order).

We then define a 'divergence' map

$$j : \mathfrak{tder}_n \to \mathfrak{T}_n$$

as follows. Let $\partial_k : A_n \to A_n$ be the linear maps defined by the identity $x = \epsilon(x)1 + \sum_{k=1}^n x_k\partial_k(x)$ (where $\epsilon : A_n \to \mathbf{k}$ is the counit map). We then set

$$j(u) := \langle\sum_{k=1}^n x_k\partial_k(u_k)\rangle.$$

[1]We set $[n] := \{1,...,n\}$.

One can show that $j$ satisfies the cocycle identity

$$j([u,v]) = u \cdot j(v) - v \cdot j(u),$$

where the action of $\mathfrak{tder}_n$ on $\mathfrak{T}_n$ is understood in the r.h.s.; $j$ is graded, so it extends to a cocycle $\mathfrak{tder}_n^\wedge \to \hat{\mathfrak{T}}_n$. The Lie algebra cocycle $j$ gives rise to the 'Jacobian' group cocycle

$$J : \mathrm{TAut}_n \to \hat{\mathfrak{T}}_n.$$

$J$ is uniquely defined by the conditions $J(\mathrm{id}) = 0$ and $(d/dt)J(e^{tx}g)_{|t=0} = j(x) + x \cdot J(g)$; as a consequence, $J$ satisfies the cocycle identity $J(h \circ g) = J(h) + h \cdot J(g)$.

The compatibility of $j, J$ with simplicial maps can be described as follows. Any partially defined $[m] \supset D_\phi \stackrel{\phi}{\to} [n]$ gives rise to a Lie algebra morphism $\mathfrak{f}_n \to \mathfrak{f}_m$, $x^{\cdot} \to x^\phi$, with $x_k^\phi := \sum_{\ell \in \phi^{-1}(k)} x_\ell$, and any $\tilde\phi$ gives rise to a morphism $\hat{\mathfrak{f}}_n \to \hat{\mathfrak{f}}_m$, $x \mapsto x^{\tilde\phi}$, with $x_k^{\tilde\phi} = \mathrm{cbh}(x_\ell | \ell \in \phi^{-1}(k))$. These morphisms give rise to linear maps $\mathfrak{T}_n \to \mathfrak{T}_m$ and $\hat{\mathfrak{T}}_n \to \hat{\mathfrak{T}}_m$. Then one can show that $j(u^\phi) = j(u)^\phi$, $J(g^\phi) = J(g)^\phi$, $j(u^{\tilde\phi}) = j(u)^{\tilde\phi}$, $J(g^{\tilde\phi}) = J(g)^{\tilde\phi}$.

We define a complex $\mathfrak{T}_1 \stackrel{\delta}{\to} \mathfrak{T}_2 \stackrel{\delta}{\to} \mathfrak{T}_3 ...$ by $f(x_1) \mapsto f(x_1+x_2) - f(x_1) - f(x_2) = f^{12} - f^1 - f^2$, $f(x_1,x_2) \mapsto f(x_1+x_2,x_3) - f(x_1,x_2+x_3) - f(x_2,x_3) + f(x_1,x_2) = f^{12,3} - f^{1,23} - f^{2,3} + f^{1,2}$, etc. It is proved in [AT] that this complex is acyclic in degree 2 (the degree of $\mathfrak{T}_i$ is $i$). The kernel of $\mathfrak{T}_1 \stackrel{\delta}{\to} \mathfrak{T}_2$ is 1-dimensional, spanned by the class of $x_1 \in A_1 \simeq \mathfrak{T}_1$.

We similarly define a complex $\hat{\mathfrak{T}}_1 \stackrel{\tilde\delta}{\to} \hat{\mathfrak{T}}_2 \stackrel{\tilde\delta}{\to} \hat{\mathfrak{T}}_3 ...$ by $f(x_1) \mapsto f(\log(e^{x_1}e^{x_2})) - f(x_1) - f(x_2) = f^{\widetilde{12}} - f^1 - f^2$, $f(x_1,x_2) \mapsto f(\log(e^{x_1}e^{x_2}), x_3) - f(x_1, \log(e^{x_2}e^{x_3})) - f(x_2,x_3) + f(x_1,x_2)$. It has a decreasing filtration by the degree, and its associated graded is the above complex, so the complex $\hat{\mathfrak{T}}_1 \stackrel{\tilde\delta}{\to} ...$ is again acyclic in degree 2. Since $\log(e^{x_1}e^{x_2}) - x_1 - x_2$ is a sum of brackets, $\mathrm{Ker}(\hat{\mathfrak{T}}_1 \stackrel{\tilde\delta}{\to} \hat{\mathfrak{T}}_2)$ is again 1-dimensional, spanned by the class of $x_1 \in A_1^\wedge \simeq \hat{\mathfrak{T}}_1$.

1.2. **Braid groups and Lie algebras of infinitesimal braids.** Let $\mathrm{B}_n$ be the braid group of order $n$. $\mathrm{B}_n$ may be viewed as $\pi_1(X_n/S_n, S_n p)$, where $X_n = \{(z_1,...,z_n) \in \mathbb{C}^n | z_i \neq z_i$ if $i \neq j\}$ and $S_n p$ is the $S_n$-orbit of the set $p = \{(z_1,...,z_n) | z_i \in \mathbb{R}, z_1 < ... < z_n\}$. The fibration $X_n \to X_n/S_n$ gives rise to the morphism $\mathrm{B}_n \to S_n$, and the pure braid group $\mathrm{PB}_n$ is defined as $\mathrm{Ker}(\mathrm{B}_n \to S_n)$, so we have an exact sequence $1 \to \mathrm{PB}_n \to \mathrm{B}_n \to S_n \to 1$; also $\mathrm{PB}_n = \pi_1(X_n, p)$.

We recall the Artin presentation of $\mathrm{B}_n$: generators are $\sigma_1, ..., \sigma_{n-1}$, and relations are given by

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (i = 1,...,n-2), \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1.$$

We also recall the Coxeter presentation of $S_n$: generators are $s_1, ..., s_{n-1}$ ($s_i$ is the permutation $(i, i+1)$) and relations are the same as those between the $\sigma_i$, with the additional relations $s_i^2 = 1$ $(i = 1,...,n-1)$. The morphism $\mathrm{B}_n \to S_n$ is then given by $\sigma_i \mapsto s_i$.

The group $\mathrm{PB}_n$ admits the following presentation. For $i < j$ $(i,j \in [n])$, set

$$x_{ij} := (\sigma_{j-2}...\sigma_i)^{-1}\sigma_{j-1}^2(\sigma_{j-2}...\sigma_i).$$

The generators $x_{ij}$ belong to $\mathrm{PB}_n$, and[2]

$$(x_{ij}x_{ik}x_{jk}, x_{ij}) = (x_{ij}x_{ik}x_{jk}, x_{ik}) = (x_{ij}x_{ik}x_{jk}, x_{jk}) = 1 \text{ for } i < j < k,$$

and

$$(x_{ij}, x_{kl}) = (x_{il}, x_{jk}) = (x_{ik}, x_{jk}x_{jl}x_{jk}^{-1}) = 1 \text{ for } i < j < k < l.$$

One proves that this constitutes a presentation of $\mathrm{PB}_n$, see Figure 1.

For any sequence $(k_1, ..., k_n)$ of integers $\geq 0$, there exists a unique morphism $\mathrm{PB}_n \to \mathrm{PB}_{k_1+...+k_n}$ consisting in replacing the first strand by $k_1$ consecutive strands, ..., the $n$th strand by $k_n$ consecutive strands. If we set $m := k_1 + ... + k_n$ and $\phi : [m] \to [n]$ is

[2] We set $(g,h) := ghg^{-1}h^{-1}$.

the map such that $\phi(k_1 + ... + k_{i-1} + [k_i]) = i$, we denote this morphism by $x \mapsto x^{\tilde{\phi}} = x^{\widetilde{1...k_1},...,\widetilde{k_1+...+k_{n-1}+1...m}}$. This morphism in explicitly given by

$$x_{ij} \mapsto \prod_{i' \in \phi^{-1}(i)}^{\nearrow} ( \prod_{j' \in \phi^{-1}(j)}^{\searrow} x_{i'j'}),$$

where $\prod^{\nearrow}, \prod^{\searrow}$ mean the product in increasing and decreasing order of the indices.

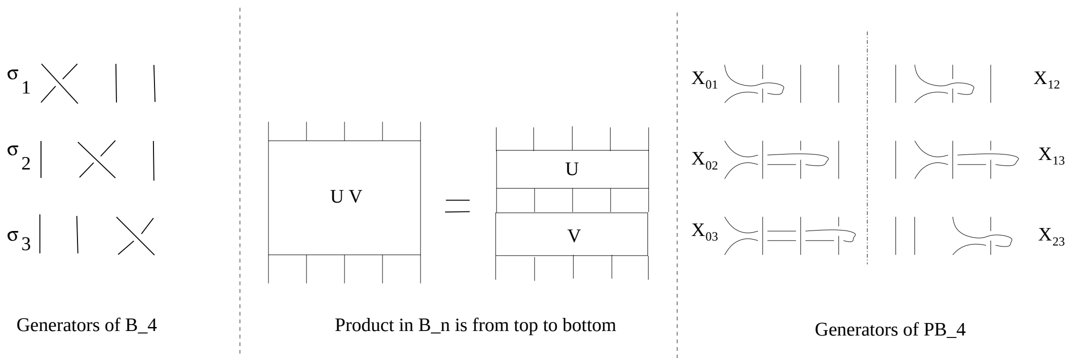


FIGURE 1.

The Lie algebra $\mathfrak{t}_n$ of infinitesimal braids is presented by generators $t_{ij}$, $i \neq j \in [n]$ and relations $t_{ji} = t_{ij}$, $[t_{ij}, t_{ik} + t_{jk}] = 0$ for $i, j, k$ distinct and $[t_{ij}, t_{kl}] = 0$ for $i, j, k, l$ distinct. For each partially defined map $[m] \supset D_\phi \stackrel{\phi}{\to} [n]$, there is a unique Lie algebra morphism $\mathfrak{t}_n \to \mathfrak{t}_m$, $x \mapsto x^\phi$ given by $t_{ij}^\phi := \sum_{i' \in \phi^{-1}(i), j' \in \phi^{-1}(j)} t_{i'j'}$ (in particular, we have an action of $S_n$ on $\mathfrak{t}_n$). We often write $x^{\phi^{-1}(1),...,\phi^{-1}(n)}$ instead of $x^\phi$. We attribute degree 1 to each of the generators $t_{ij}$, so the Lie algebra $\mathfrak{t}_n$ is positively graded; we denote by $\hat{\mathfrak{t}}_n$ its degree completion.

1.3. **The morphism** $\mathfrak{t}_{n+1} \to \mathfrak{tder}_n$**.** Let us reindex $t_{ij}$, $i \neq j \in \{0, ..., n\}$ the generators of $\mathfrak{t}_{n+1}$. One checks that there is a unique morphism ad : $\mathfrak{t}_{n+1} \to \mathfrak{tder}_n$, defined by $t_{0i} \mapsto (x_j \mapsto [x_i, x_j])$ and $t_{ij} \mapsto (x_i \mapsto [x_i, x_j], x_j \mapsto [x_j, x_i], x_k \mapsto 0$ for $k \neq i, j)$ if $i, j \neq 0$. It exponentiates to Ad : $\exp(\hat{\mathfrak{t}}_{n+1}) \to \mathrm{TAut}_n$. One checks that $j(\mathrm{ad}\, t_{ij}) = 0$, so the cocycle property implies $j(\mathrm{ad}\, x) = J(\mathrm{Ad}\, X) = 0$ for any $x \in \mathfrak{t}_{n+1}$ and $X \in \exp(\hat{\mathfrak{t}}_{n+1})$.

The morphism ad : $\mathfrak{t}_{n+1} \to \mathfrak{tder}_n$ may be interpreted as follows. The Lie subalgebra of $\mathfrak{t}_{n+1}$ generated by the elements $t_{0i}$, $i \in [n]$ identifies with $\mathfrak{f}_n$ under $x_i \mapsto t_{0i}$; it is a Lie ideal of $\mathfrak{t}_{n+1}$. Then ad : $\mathfrak{t}_{n+1} \to \mathrm{Der}(\mathfrak{f}_n)$ can be viewed as the adjoint action of $\mathfrak{t}_{n+1}$ on its Lie ideal $\mathfrak{f}_n \subset \mathfrak{t}_{n+1}$.

Note that the morphism $\mathfrak{t}_n \to \mathfrak{t}_{n+1}$, $t_{ij} \mapsto t_{ij}$ is injective, so $\mathfrak{t}_n$ may be viewed as a Lie subalgebra of $\mathfrak{t}_{n+1}$; then $\mathfrak{t}_{n+1}$ identifies with the semidirect product $\mathfrak{f}_n \rtimes_{\mathrm{ad}} \mathfrak{t}_n$.

1.4. **The morphism** $\mathrm{PB}_{n+1} \to \mathrm{TAut}_n$**.** Reindex the generators of $\mathrm{PB}_{n+1}$ as $x_{ij}$, $i < j \in \{0, ..., n\}$. Let $\mathrm{F}_n$ be the free group with generators $X_i$ ($i \in [n]$). Then: (a) the morphism $\mathrm{F}_n \to \mathrm{PB}_{n+1}$, $X_i \mapsto x_{0i}$, is injective; (b) $\mathrm{F}_n$ is a normal subgroup in $\mathrm{PB}_{n+1}$. This implies that we have an action $\mathrm{Ad} : \mathrm{PB}_{n+1} \to \mathrm{Aut}(\mathrm{F}_n)$ of $\mathrm{PB}_n$ by automorphisms of $\mathrm{F}_n$.

This action can be made explicit as follows: if $i > 0$, then

$$\mathrm{Ad}(x_{0i})(X_j) = X_i X_j X_i^{-1},$$

and if $0 < i < j$, then

$$\mathrm{Ad}(x_{ij})(X_i) = X_j^{-1} X_i X_j, \quad \mathrm{Ad}(x_{ij})(X_j) = (X_i X_j)^{-1} X_j (X_i X_j),$$
$$\mathrm{Ad}(x_{ij})(X_k) = X_k \quad \text{for} \quad k < i \quad \text{or} \quad k > j,$$
$$\mathrm{Ad}(x_{ij})(X_k) = (X_j^{-1} X_i^{-1} X_j X_i) X_k (X_j^{-1} X_i^{-1} X_j X_i)^{-1} \quad \text{for} \quad i < k < j.$$

This extends to an action of $\mathrm{PB}_{n+1}$ by automorphisms of $\mathrm{F}_n(\mathbf{k})$. Using the isomorphism $\mathrm{F}_n(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_n)$ given by $X_i \mapsto e^{x_i}$, we therefore obtain a morphism $\mathrm{PB}_n \to \mathrm{Aut}(\hat{\mathfrak{f}}_n)$. Its image is contained in $\mathrm{TAut}_n$ (since $\mathrm{Ad}\, x_{ij}$ belongs to this subgroup and the elements $x_{ij}$ generate $\mathrm{PB}_n$), and since $\mathrm{TAut}_n$ is prounipotent, the universal property of Malcev completions implies that Ad extends to a morphism $\mathrm{Ad} : \mathrm{PB}_n(\mathbf{k}) \to \mathrm{TAut}_n$.

**Lemma 1.1.** *For any $g \in \mathrm{PB}_{n+1}(\mathbf{k})$, $J(\mathrm{Ad}\, g) = 0$.*

*Proof.* It suffices to show that $J(\mathrm{Ad}\, x_{ij}) = 0$. For any $u \in \mathrm{F}_n(\mathbf{k})$, $J(\mathrm{Inn}\, u) = 0$ (where $\mathrm{Inn}\, u$ is $v \mapsto uvu^{-1}$) and $\mathrm{Ad}(x_{0i}) = \mathrm{Inn}(X_i)$, so it suffices to prove that $J(\mathrm{Inn}\, X_j \circ \mathrm{Ad}\, x_{ij}) = 0$ for $0 < i < j$. Let $\theta_{ij} := \mathrm{Inn}\, X_j \circ \mathrm{Ad}\, x_{ij}$, then $\theta_{ij} : X_i \mapsto X_i$, $X_j \mapsto X_i^{-1} X_j X_i$, $X_k \mapsto X_j X_k X_j^{-1}$ for $k < i$ or $k > j$, $X_k \mapsto (X_i^{-1} X_j X_i) X_k (X_i^{-1} X_j X_i)^{-1}$ for $i < k < j$.

Let $\mathfrak{u} \subset \mathfrak{tder}_n^{\wedge}$ be the subspace of all elements $[\![a_1, ..., a_n]\!]$, where $a_j \in \mathbf{k} x_i$, $a_i = 0$, and for $k \neq i, j$, $a_k \in \hat{\mathfrak{f}}_n$ has the form $a_k(x_i, x_j)$ ($a_k \in \hat{\mathfrak{f}}_2$). This is a Lie subalgebra in $\mathfrak{tder}_n^{\wedge}$, so exp maps it bijectively to a subgroup of $\mathrm{TAut}_n$. One checks that $\exp(\mathfrak{u}) \subset U$, where $U \subset \mathrm{TAut}_n$ is the subspace of all $[\![U_1, ..., U_n]\!]$, where $U_j \in \{e^{\lambda x_i}, \lambda \in \mathbf{k}\}$, $U_i = 1$, and for $k \neq i, j$, $U_k$ has the form $U_k(x_i, x_j)$ ($U_k \in \exp(\hat{\mathfrak{f}}_2)$), and that $U$ is an algebraic subgroup of $\mathrm{TAut}_n$. Therefore $\mathfrak{u} \subset \mathrm{Lie}(U)$. On the other hand, one checks that $\mathfrak{u}$ coincides with the tangent subspace of $U$ at the origin, so $\mathfrak{u} = \mathrm{Lie}(U)$. It follows that log takes $U$ to $\mathfrak{u}$.

All this implies that $\log \theta_{ij}$ has the form $[\![a_1, ..., a_n]\!]$, where $a_i = 0$, $a_j = -x_i$ and for $k \neq i, j$, $a_k \in \hat{\mathfrak{f}}_n$ has the form $a_k(x_i, x_j)$. Then $j(\log \theta_{ij}) = 0$, hence $J(\theta_{ij}) = 0$, as wanted. □

Note that the quotient group $\mathrm{PB}_{n+1} / \mathrm{F}_n$ identifies with $\mathrm{PB}_n$ under $x_{ij} \mapsto x_{ij}$ for $0 < i < j$, $x_{0i} \mapsto 1$. We then have an exact sequence $1 \to \mathrm{F}_n \to \mathrm{PB}_{n+1} \to \mathrm{PB}_n \to 1$. Moreover, this exact sequence admits the splitting $\mathrm{PB}_n \to \mathrm{PB}_{n+1}$, $x_{ij} \mapsto x_{ij}$. It follows that $\mathrm{PB}_{n+1}$ identifies with the semidirect product $\mathrm{F}_n \rtimes_{\mathrm{Ad}} \mathrm{PB}_n$.

**Remark 1.2.** We will rename $x, y$ (resp., $x, y, z$, $X, Y$, $X, Y, Z$) the generators $x_1, x_2$ (resp., $x_1, x_2, x_3$, $X, Y$, $X, Y, Z$) of $\hat{\mathfrak{f}}_2$ (resp., $\hat{\mathfrak{f}}_3$, $\mathrm{F}_2$, $\mathrm{F}_3$).

## 2. The main results

2.1. **The map $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$.** Let $\hat{\mathfrak{f}}_2$ be the topologically free Lie algebra generated by $x, y$. Let $\mathrm{F}_2$ be the free group with generators $X, Y$ and let $\mathrm{F}_2(\mathbf{k})$ be its prounipotent completion; we have an identification $\mathrm{F}_2(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_2)$, induced by the morphism $\mathrm{F}_2 \to \exp(\hat{\mathfrak{f}}_2)$ given by $X \mapsto e^x, Y \mapsto e^y$.

The set of solutions of the Kashiwara–Vergne equations is (see [KV, AT])[3] [4] [5]

$$\mathrm{SolKV}(\mathbf{k}) := \{\mu \in \mathrm{Iso}(\mathrm{F}_2(\mathbf{k}), \exp(\hat{\mathfrak{f}}_2)) | \mu(X) \sim e^x, \mu(Y) \sim e^y, \mu(XY) = e^{x+y},$$
$$\text{and } \exists r \in u^2 \mathbf{k}[[u]] | J(\mu) = \langle r(x+y) - r(x) - r(y) \rangle\}.$$

Here $\mu$ gives rise to an element of $\mathrm{TAut}_2$ (using $\mathrm{F}_2(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_2)$) and $J(\mu)$ is its Jacobian. As the kernel of $\mathfrak{T}_1 \to \mathfrak{T}_2$ is equal to $\mathbf{k} u$, $r$ is uniquely determined by $\mu \in \mathrm{SolKV}(\mathbf{k})$, so we

[3]For $g, h$ in a prounipotent group $G$ or its Lie algebra, we use the notation $g \sim h$ for '$g$ is conjugated to $h$', i.e., $g = khk^{-1}$ for some $k \in G$.

[4]If $\Gamma$ is a finitely generated group, we denote by $\Gamma(\mathbf{k})$ its prounipotent (of Malcev) completion. There is a group morphism $\Gamma \to \Gamma(\mathbf{k})$ with the universal property that any group morphism $\Gamma \to U$, with $U$ prounipotent, extends uniquely to a morphism $\Gamma(\mathbf{k}) \to U$ of algebraic groups.

[5]The definition given here is equivalent to that of [AT] as $\mathfrak{T}_1 \to \mathfrak{T}_2 \to \mathfrak{T}_3$ is acyclic.

define a map $\mathrm{Duf} : \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$, $\mu \mapsto r = \mathrm{Duf}(\mu)$; we will call $r$ the Duflo formal series of $\mu$.

The set of associators with coupling constant 1 is

$M_1(\mathbf{k}) := \{\Phi(t_{12}, t_{23}) \in \exp(\hat{\mathfrak{t}}_3) | \Phi^{3,2,1} = \Phi^{-1}, e^{t_{23}/2}\Phi^{1,2,3}e^{t_{12}/2}\Phi^{3,1,2}e^{t_{31}/2}\Phi^{2,3,1} = e^{(t_{12}+t_{23}+t_{31})/2}$, $\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{1,2,34}\Phi^{12,3,4}\}$.

**Theorem 2.1.** *There is a unique map* $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$, $\Phi \mapsto \mu_\Phi$, *such that*[6]

$$\mu_\Phi(X) = \Phi(x, -x-y)e^x\Phi(x, -x-y)^{-1}, \mu_\Phi(Y) = e^{-(x+y)/2}\Phi(y, -x-y) \cdot e^y \cdot (\mathit{same})^{-1}.$$

The Jacobian of $\mu_\Phi$ can be computed as follows. In [DT, E] (see also [Ih]), it was proved[7] that for any $\Phi(a, b) \in M_1(\mathbf{k})$, there exists a formal series $\Gamma_\Phi(u) = e^{\sum_{n\geq 2}(-1)^n\zeta_\Phi(n)u^n/n}$, such that

$$(1 + b\partial_b\Phi(a, b))^{ab} = \frac{\Gamma_\Phi(\overline{a} + \overline{b})}{\Gamma_\Phi(\overline{a})\Gamma_\Phi(\overline{b})}, \tag{6}$$

where $\partial_b\Phi(a, b)$ is defined as above and $x \mapsto x^{ab}$ is the abelianization morphism $\mathbf{k}\langle\langle a, b\rangle\rangle \to \mathbf{k}[[\overline{a}, \overline{b}]]$

The values of the $\zeta_\Phi(n)$ for $n$ even are independent of $\Phi$, given by $-\frac{1}{2}(\frac{u}{e^u-1} - 1 + \frac{u}{2}) = \sum_{n\geq 1}\zeta_\Phi(2n)u^{2n}$, so they are related to Bernoulli numbers by $\zeta_\Phi(2n) = -\frac{1}{2}\frac{B_{2n}}{(2n)!}$ for $n \geq 1$ (we have $\zeta_\Phi(2) = -1/24$, $\zeta_\Phi(4) = 1/1440$, etc.)

**Proposition 2.2.** $J(\mu_\Phi) = \langle \log\Gamma_\Phi(x) + \log\Gamma_\Phi(y) - \log\Gamma_\Phi(x+y)\rangle$, *so* $\mathrm{Duf}(\mu_\Phi) = -\log\Gamma_\Phi$.

2.2. **Torsor aspects.** We set

$$\mathrm{KV}(\mathbf{k}) := \{\alpha \in \mathrm{Aut}(\mathrm{F}_2(\mathbf{k})) | \alpha(X) \sim X, \alpha(Y) \sim Y, \alpha(XY) = XY,$$
$$\text{and } \exists\sigma \in u^2\mathbf{k}[[u]] | J(\alpha) = \langle\sigma(\log(e^xe^y)) - \sigma(x) - \sigma(y)\rangle\}$$

and

$$\mathrm{KRV}(\mathbf{k}) := \{a \in \mathrm{Aut}(\hat{\mathfrak{f}}_2) | a(x) \sim x, a(y) \sim y, a(x+y) = x+y,$$
$$\text{and } \exists s \in u^2\mathbf{k}[[u]] | J(a) = \langle s(x+y) - s(x) - s(y)\rangle\}.$$

Here $\alpha, a$ give rise to elements of $\mathrm{TAut}_2$ (using $\mathrm{F}_2(\mathbf{k}) \simeq \exp(\hat{\mathfrak{f}}_2)$) and $J(\alpha), J(a)$ are their Jacobians. As before, we will denote $\mathrm{Duf} : \mathrm{KV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$, $\mathrm{KRV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ the maps $\alpha \mapsto \sigma$, $a \mapsto s$.

**Proposition 2.3.** $\mathrm{KV}(\mathbf{k})$ *and* $\mathrm{KRV}(\mathbf{k})$ *are groups.* $\mathrm{SolKV}(\mathbf{k})$ *is a torsor under the commuting left action of* $\mathrm{KV}(\mathbf{k})$ *and right action of* $\mathrm{KRV}(\mathbf{k})$ *given by* $(\alpha, \mu) \mapsto \mu \circ \alpha^{-1}$ *and* $(\mu, a) \mapsto a^{-1} \circ \mu$.

---

[6]If $G$ is a prounipotent group, we use the notation $g \cdot h \cdot (\mathit{same})^{-1}$ for $ghg^{-1}$ for if $g \in G$ and $h \in G$ or $\mathrm{Lie}(G)$.

[7]The key ingredient in the proof of this result is the statement that the image of $\mathfrak{grt}_1$ in $\mathfrak{f}_2'/\mathfrak{f}_2''$ is spanned by the classes of the Drinfeld generators. This statement also follows from Theorem 4.1 in [AT]; indeed, one sees easily that the diagram

$$\begin{array}{ccccc}
\mathfrak{f}_2 & \overset{\psi\mapsto\langle a\partial_a\psi\rangle}{\to} & \mathfrak{T}_2 & \overset{\phi\mapsto\phi^{ab}}{\to} & \mathbf{k}[\bar{a}, \bar{b}] \\
\uparrow & & & & \uparrow \\
\mathfrak{f}_2' & & \to & & \mathfrak{f}_2'/\mathfrak{f}_2'' = \bar{a}\bar{b}\mathbf{k}[\bar{a}, \bar{b}] \\
\uparrow & & & & \uparrow \\
\mathfrak{grt}_1 & & \to & & \mathfrak{grt}_1/\mathfrak{grt}_1'
\end{array}$$

commutes (the upper part follows from the fact that $\mathfrak{f}_2'$ is freely generated by the $(\mathrm{ad}\, a)^k(\mathrm{ad}\, b)^l([a, b])$ and the bottom part from $\mathfrak{grt}_1' \subset \mathfrak{f}_2''$); Theorem 4.1 in [AT] implies that the image of $\mathfrak{grt}_1 \to \mathfrak{T}_2$ is spanned by the images of the Drinfeld generators; it follows that the same is true of the image of $\mathfrak{grt}_1 \to \mathfrak{f}_2'/\mathfrak{f}_2''$.

In particular, any element of $\mathrm{SolKV}(\mathbf{k})$ gives rise to an isomorphism $\mathfrak{kv} \to \mathfrak{krv}$ between the Lie algebras of these groups, whose associated graded is the canonical identification $\mathrm{gr}(\mathfrak{kv}) \simeq \mathfrak{krv}$.

The prounipotent radical of the Grothendieck-Teichmüller group is

$$\mathrm{GT}_1(\mathbf{k}) = \{f(X,Y) \in \mathrm{F}_2(\mathbf{k}) | f(Y,X) = f(X,Y)^{-1}, f(X,Y)f(Y^{-1}X^{-1},X)f(Y,Y^{-1}X^{-1}) = 1,$$
$$f(x_{23},x_{34})f(x_{12}x_{13},x_{24}x_{34})f(x_{12},x_{23}) = f(x_{12},x_{23}x_{24})f(x_{13}x_{23},x_{34})\}$$

(the last equation is in $\mathrm{PB}_4(\mathbf{k})$) with product $(f_1 * f_2)(X,Y) = f_1(f_2(X,Y)Xf_2(X,Y)^{-1},Y)f_2(X,Y)$. Its graded version is

$$\mathrm{GRT}_1(\mathbf{k}) = \{g(t_{12},t_{23}) \in \exp(\hat{\mathfrak{t}}_3) | g^{3,2,1} = g^{-1}, g(A,C)Ag(A,C)^{-1} + g(B,C)Bg(B,C)^{-1} + C = 0$$
$$\text{for } A+B+C=0, g^{1,2,3}g^{3,1,2}g^{2,3,1} = 1, g^{2,3,4}g^{1,23,4}g^{1,2,3} = g^{1,2,34}g^{12,3,4}\}$$

with product $(g_1 * g_2)(a,b) = g_1(g_2(a,b)ag_2(a,b)^{-1},b)g_2(a,b)$ (we set $a := t_{12}, b := t_{23}$).

**Proposition 2.4.** *(see* [Dr]*) $M_1(\mathbf{k})$ is a torsor under the commuting left action of $\mathrm{GT}_1(\mathbf{k})$ and right action of $\mathrm{GRT}_1(\mathbf{k})$ by $(f,\Phi) \mapsto (f * \Phi)(a,b) := f(\Phi(a,b)e^a\Phi(a,b)^{-1}, e^b)\Phi(a,b)$ and $(\Phi,g) \mapsto (\Phi * g)(a,b) := \Phi(g(a,b)ag(a,b)^{-1},b)g(a,b)$.*

The following Theorem 2.5 and Proposition 2.6 express torsor properties of the map $\Phi \mapsto \mu_\Phi$.

**Theorem 2.5.** *There are unique group morphisms $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$, $f(X,Y) \mapsto \alpha_f^{-1}$, where*

$$\alpha_f(X) = f(X,Y^{-1}X^{-1})Xf(X,Y^{-1}X^{-1})^{-1}, \alpha_f(Y) = f(Y,Y^{-1}X^{-1})Yf(Y,Y^{-1}X^{-1})^{-1},$$

*and $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{KRV}_1(\mathbf{k})$, $g(a,b) \mapsto a_g^{-1}$, where*

$$a_g(x) = g(x,-x-y)xg(x,-x-y)^{-1}, a_g(y) = g(y,-x-y)yg(y,-x-y)^{-1}.$$

*These group morphisms are compatible with the map $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$, which is therefore a morphism of torsors.*

**Proposition 2.6.** *We have a commuting diagram of torsors*

$$\begin{array}{ccc} M_1(\mathbf{k}) & \stackrel{\Phi\mapsto\mu_\Phi}{\rightarrow} & \mathrm{SolKV}(\mathbf{k}) \\ {\scriptstyle \Phi\mapsto\log\Gamma_\Phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Duf}} \\ \{r \in u^2\mathbf{k}[[u]] | r_{ev}(u) = -\frac{u^2}{24} + \frac{u^4}{1440}...\} & \stackrel{(-1)\times -}{\hookrightarrow} & u^2\mathbf{k}[[u]] \end{array}$$

*where $r_{ev}(u)$ is the even part of $r(u)$, and the spaces in the lower line are viewed as affine spaces.*

2.3. **Analytic aspects.** Let us recall the original form of the KV conjecture. Let $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$.

**Conjecture 2.7.** *(*[KV]*) For any finite dimensional $\mathbf{k}$-Lie algebra $\mathfrak{g}$, there exists a pair of Lie series $A(x,y), B(x,y) \in \hat{\mathfrak{f}}_2$, such that:*

*(KV1) $x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})(A(x,y)) + (e^{\operatorname{ad} y} - 1)(B(x,y))$;*

*(KV2) $A, B$ give convergent power series at the neighborhood of $(0,0) \in \mathfrak{g}^2$;*

*(KV3) $\mathrm{tr}_{\mathfrak{g}}((\operatorname{ad} x)\partial_x A + (\operatorname{ad} y)\partial_y B) = \frac{1}{2}\mathrm{tr}_{\mathfrak{g}}(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x}-1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y}-1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z}-1} - 1)$ (identity of analytic functions on $\mathfrak{g}^2$ near the origin), where $z = \log e^x e^y$ and for $(x,y) \in \mathfrak{g}^2$, $(\partial_x A)(x,y) \in \mathrm{End}(\mathfrak{g})$ is $a \mapsto \frac{d}{dt}_{|t=0} A(x+ta,y)$, $(\partial_y B)(x,y)(a) = \frac{d}{dt}_{|t=0} B(x,y+ta)$.*

According to [AT], there is a unique map $\kappa : \mathrm{TAut}_2 \to \mathfrak{tder}_2$, where $\kappa(g) := \ell - g\ell g^{-1}$, and $\ell \in \mathrm{Der}(\hat{\mathfrak{f}}_2)$ is the 'grading' derivation $\ell(x_i) = x_i$. It is proved in [AT] that if $\mu \in \mathrm{SolKV}(\mathbf{k})$, and $(A,B)$ are such that $-\kappa(\mu^{-1}) = [\![A,B]\!]$, then (KV1) and (KV3) hold as identities between formal series for any $\mathfrak{g}$, where in (KV3) the formal series $\frac{1}{2}\frac{t}{e^t-1}$ is replaced by $r_\mu(t)$.

Let $\Phi_{\mathrm{KZ}}(a,b) \in \exp(\hat{\mathfrak{f}}_2)$ be the KZ associator, and $\tilde{\Phi}_{\mathrm{KZ}}(a,b) := \Phi_{\mathrm{KZ}}(\frac{a}{2\pi\mathrm{i}}, \frac{b}{2\pi\mathrm{i}})$; recall that $\tilde{\Phi}_{\mathrm{KZ}}$ is the renormalized holonomy from 0 to 1 of $G'(t) = \frac{1}{2\pi\mathrm{i}}(\frac{a}{t} + \frac{b}{t-1})G(t)$, and $\tilde{\Phi}_{\mathrm{KZ}} \in M_1(\mathbb{C})$. Set $\mu_{\mathrm{KZ}} := \mu_{\tilde{\Phi}_{\mathrm{KZ}}}$ and $u_{\mathrm{KZ}} = [\![A_{\mathrm{KZ}}, B_{\mathrm{KZ}}]\!] := -\kappa(\mu_{\mathrm{KZ}}^{-1})$.

Let $(A_{\mathbb{R}}, B_{\mathbb{R}})$ be defined as the real parts of $(A_{\mathrm{KZ}}, B_{\mathrm{KZ}})$ (w.r.t. the natural real structure of $\hat{\mathfrak{f}}_2$). Then:

**Theorem 2.8.** *1) $(A_{\mathbb{R}}, B_{\mathbb{R}})$ satisfies (KV1), (KV2) and (KV3) for any finite dimensional Lie algebra $\mathfrak{g}$ and is therefore a universal solution of the KV conjecture.*

*2) For any $s \in \mathbb{R}$, $(A_s, B_s) := (A_{\mathbb{R}} + s(\log(e^x e^y) - x), B_{\mathbb{R}} + s(\log(e^x e^y) - y))$ is a universal solution of the KV conjecture.*

*3) When $s = -1/4$, we have $(A_s(x,y), B_s(x,y)) = (B_s(-y,-x), A_s(-y,-x))$.*

Of course, the main new result here is the analyticity statement (KV2).

2.4. **Organization of the proofs.** We construct the isomorphisms $\tilde{\mu}_\Phi^O$ and $\mu_\Phi^O$ in Section 3. In Section 4, we prove the identity relating $\mu_O$ and $\mu_{O^{(i)}}$. We then prove Theorem 2.1 and Proposition 2.2 in Section 5. In Section 6, we prove Proposition 2.3, Theorem 2.5 and Proposition 2.6. Section 7 is devoted to a direct proof of the properties of $\alpha_f$. In Section 8, we compute the Jacobians of $\mu_\Phi^O$ and $\alpha_f^O$ and in Section 9, we prove the analytic Theorem 2.8. Appendix A is devoted to results on centralizers in $\mathfrak{t}_n$ and $\mathrm{PB}_n(\mathbf{k})$.

## 3. Associators and isomorphisms of free groups

3.1. **The categories PaB, PaCD.** In [B], Bar-Natan introduced the category **PaB** of parenthesized braids. Its set of objects is the set of pairs $O = (n, P)$, where $n$ in an integer $\geq 0$ and $P$ is a parenthesization of the word $\bullet ... \bullet$ ($n$ letters); alternatively, $P$ is a planar binary tree with $n$ leaves (we will set $|O| = n$). The object with $n = 0$ is denoted $\mathbf{1}$. The morphisms are defined by $\mathbf{PaB}(O, O') = \emptyset$ if $|O| \neq |O'|$, and $= \mathrm{B}_n$ if $|O| = |O'| = n$; the composition is then defined using the product in $\mathrm{B}_n$.

**PaB** is a braided monoidal category (see e.g. [CE]), where the tensor product of objects is $(n,P) \otimes (n',P') := (n+n', P * P')$ (where $P * P'$ is the concatenation of parenthesized words, e.g. for $P = \bullet\bullet$ and $P' = (\bullet\bullet)\bullet$, $P * P' = (\bullet\bullet)((\bullet\bullet)\bullet)$). The tensor product of morphisms $\mathbf{PaB}(O_1, O_1') \times \mathbf{PaB}(O_2, O_2') \to \mathbf{PaB}(O_1 \otimes O_2, O_1' \otimes O_2')$ is induced by the juxtaposition of braids $\mathrm{B}_{|O_1|} \times \mathrm{B}_{|O_2|} \to \mathrm{B}_{|O_1|+|O_2|}$ (the group morphism $(\sigma_i, e) \mapsto \sigma_i$, $(e, \sigma_j) \mapsto \sigma_{j+|O_1|}$). The braiding $\beta_{O,O'} \in \mathbf{PaB}(O \otimes O', O' \otimes O)$ is the braid $\sigma_{n,n'} \in \mathrm{B}_{n+n'}$ where the $n$ first strands are globally exchanged with the $n'$ last strands (see Figure 2); we have $\sigma_{n,n'} = (\sigma_n ... \sigma_1)(\sigma_{n+1} ... \sigma_2) ... (\sigma_{n+n'-1} ... \sigma_{n'})$ (where $n = |O|$, $n' = |O'|$). Finally, the associativity constraint $a_{O,O',O''} \in \mathbf{PaB}((O \otimes O') \otimes O'', O \otimes (O' \otimes O''))$ corresponds to the trivial braid $e \in B_{|O|+|O'|+|O''|}$.

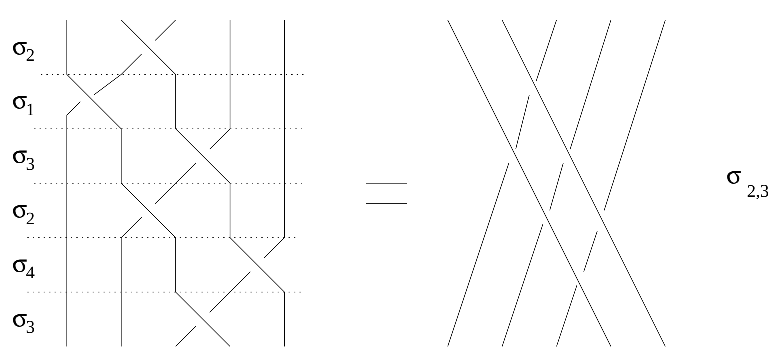


FIGURE 2.

Moreover, the pair $(\mathbf{PaB},\bullet)$ is universal for pairs $(\mathcal{C},M)$ of a braided monoidal category and an object, i.e., for each such pair, there exists a unique tensor functor $\mathbf{PaB}\to\mathcal{C}$ taking $\bullet$ to $M$.

Bar-Natan introduced another category $\mathbf{PaCD}$, which we will describe as follows. Its set of objects is the same as that of $\mathbf{PaB}$, and $\mathbf{PaB}(O,O')=\emptyset$ if $|O|\neq|O'|$, $=\exp(\hat{\mathfrak{t}}_n)\rtimes S_n$ if $|O|=|O'|=n$. We define the tensor product as above at the level of objects, and by the juxtaposition map $(\exp\hat{\mathfrak{t}}_n\rtimes S_n)\times(\exp\hat{\mathfrak{t}}_n\rtimes S_n)\to\exp\hat{\mathfrak{t}}_{n+n'}\rtimes S_{n+n'}$ (the group morphism induced by $((t_{ij},1),1)\mapsto t_{ij}$, $((1,s_i),1)\mapsto s_i$, $(1,(t_{ij},1))\mapsto t_{n+i,n+j}$, $(1,(1,s_i))\mapsto s_{n+i}$) at the level of morphisms.

Any $\Phi\in M_1(\mathbf{k})$ gives rise to a structure of braided monoidal category on $\mathbf{PaCD}$ (and therefore to a tensor functor $\mathbf{PaB}\to\mathbf{PaCD}$, which is the identity at the level of objects) as follows: $\beta_{O,O'}=e^{\sum_{i=1}^{n}\sum_{j=n+1}^{n+n'}t_{ij}/2}s_{n,n'}$, where $n=|O|,n'=|O'|$, and $s_{n,n'}\in S_{n+n'}$ is given by $s_{n,n'}(i)=n'+i$ for $i\in[n]$, $s_{n,n'}(n+i)=i$ for $i\in[n']$, and $a_{O,O',O''}=\Phi(t_{12},t_{23})^{1\dots n,n+1\dots n+n',n+n'+1\dots n+n'+n''}$ for $n=|O|$, $n'=|O'|$, $n''=|O''|$.

3.2. **Morphisms** $\mathrm{B}_n\to\exp(\hat{\mathfrak{t}}_n)\rtimes S_n$**,** $\mathrm{PB}_n\to\exp(\hat{\mathfrak{t}}_n)$**.** Fix $\Phi\in M_1(\mathbf{k})$. It gives rise to a functor $F_\Phi:\mathbf{PaB}\to\mathbf{PaCD}$, so for any $n\geq 1$ and any $O\in\mathrm{Ob}(\mathbf{PaB})$, $|O|=n$, we get a group morphism

$$F_\Phi(O)=\tilde{\mu}_O:\mathrm{B}_n\simeq\mathbf{PaB}(O)\to\mathbf{PaCD}(O)=\exp(\hat{\mathfrak{t}}_n)\rtimes S_n,$$

such that $\begin{array}{ccc}\mathrm{B}_n & \stackrel{\tilde{\mu}_O}{\to} & \exp(\hat{\mathfrak{t}}_n)\rtimes S_n\\ \searrow & & \swarrow\\ & S_n & \end{array}$ commutes. It follows that $\tilde{\mu}_O$ restricts to a morphism

$$\tilde{\mu}_O:\mathrm{PB}_n\to\exp(\hat{\mathfrak{t}}_n).$$

Let us show that the various $\tilde{\mu}_O$ are are all conjugated to each other. Let $\mathrm{can}_{O,O'}\in\mathbf{PaB}(O,O')$ correspond to $e\in\mathrm{B}_n$. Then $\mathrm{can}_{O',O''}\circ\mathrm{can}_{O,O'}=\mathrm{can}_{O,O''}$. Moreover, if we denote by $\sigma_O:\mathrm{B}_n\to\mathbf{PaB}(O)$ the canonical identification, then $\sigma_{O'}(b)=\mathrm{can}_{O,O'}\circ\sigma_O(b)\circ\mathrm{can}_{O,O'}^{-1}$. Let us set $\Phi_{O,O'}:=F_\Phi(\mathrm{can}_{O,O'})$. Then:
1) $\Phi_{O,O'}\in\exp(\hat{\mathfrak{t}}_n)$, $\Phi_{O',O''}\Phi_{O,O'}=\Phi_{O,O'}$;
2) $\tilde{\mu}_{O'}(b)=\Phi_{O,O'}\tilde{\mu}_O(b)\Phi_{O,O'}^{-1}$.
If $O=\bullet(...(\bullet\bullet))$ is the 'right parenthesization', the explicit formula for $\tilde{\mu}_O$ is

$$\tilde{\mu}_O(\sigma_i)=\Phi^{i,i+1,i+2\dots n}e^{t_{i,i+1}/2}s_i(\Phi^{i,i+1,i+2\dots n})^{-1},\quad i=0,...,n-1.$$

The morphisms $\tilde{\mu}_O$ extend to isomorphisms between prounipotent completions as follows. The prounipotent completion of $\mathrm{B}_n$ relative to $\mathrm{B}_n\to S_n$ will be denoted $\mathrm{B}_n(\mathbf{k},S_n)$; it may be constructed as follows: $\mathrm{B}_n$ acts by automorphisms of $\mathrm{PB}_n$, hence of $\mathrm{PB}_n(\mathbf{k})$; $\mathrm{B}_n(\mathbf{k},S_n)$ fits in an exact sequence $1\to\mathrm{PB}_n(\mathbf{k})\to\mathrm{B}_n(\mathbf{k},S_n)\to S_n\to 1$ and identifies with the quotient of the semidirect product $\mathrm{PB}_n(\mathbf{k})\rtimes\mathrm{B}_n$ by the image of the morphism $\mathrm{PB}_n\to\mathrm{PB}_n(\mathbf{k})\rtimes\mathrm{B}_n$, $g\mapsto(g^{-1},g)$ (which is a normal subgroup). Then the morphisms $\tilde{\mu}_O$ give rise to isomorphisms

$$\begin{array}{ccc}\mathrm{PB}_n(\mathbf{k}) & \stackrel{\sim}{\to} & \exp(\hat{\mathfrak{t}}_n)\\ \downarrow & & \downarrow\\ \mathrm{B}_n(\mathbf{k},S_n) & \stackrel{\sim}{\to} & \exp(\hat{\mathfrak{t}}_n)\rtimes S_n\end{array}$$

When $\Phi$ is the KZ associator (with coupling constant $2\pi\mathrm{i}$), these isomorphisms are given by Sullivan's theory of minimal models applied to the configuration space of $n$ points in the complex plane (which computes all the rational homotopy groups of a simply-connected Kaehler manifold, but only the Malcev completion of its fundamental group in the non-simply-connected case, whence the name '1-formality').

3.3. **Restriction to free groups.** Renumber $x_{ij}$, $i < j \in \{0, ..., n\}$ and $t_{ij}$, $i \neq j \in \{0, ..., n\}$ the generators for $\mathrm{PB}_{n+1}$ and $\mathfrak{t}_{n+1}$. Recall that $\mathrm{PB}_{n+1}$ contains the free group with $n$ generators $\mathrm{F}_n = \langle x_{01}, ..., x_{0,n} \rangle$ as a normal subgroup. Similarly, $\mathfrak{t}_{n+1}$ contains the free Lie algebra with $n$ generators $\mathfrak{f}_n = \mathrm{Lie}(t_{01}, ..., t_{0,n})$. For coherence of notation with the previous sections, we will set $X_i := x_{0i}$, $x_i := t_{0i}$.

**Proposition 3.1.** *For any* $O \in \mathrm{Ob}(\mathbf{PaB})$ *with* $|O| = n+1$*, the morphism* $\tilde{\mu}_O$ *restricts to a morphism* $\mu_O : \mathrm{F}_n \to \exp(\hat{\mathfrak{f}}_n)$*, which extends to an isomorphism* $\mu_O : \mathrm{F}_n(\mathbf{k}) \to \exp(\hat{\mathfrak{f}}_n)$*. The composition of* $\mu_O$ *with the isomorphism* $\exp(\hat{\mathfrak{f}}_n) \to \mathrm{F}_n(\mathbf{k})$, $\exp(x_i) \mapsto X_i$*, is a tangential automorphism of* $\exp(\hat{\mathfrak{f}}_n)$*, i.e., an element of* $\mathrm{TAut}_n$.

*Proof.* Let us first treat the case of $\mu_n := \mu_{\bullet(...(\bullet\bullet))}$. As $x_{0i} = (\sigma_{i-2}...\sigma_0)^{-1}\sigma_{i-1}^2(\sigma_{i-2}...\sigma_0)$, we have $\mu_n(x_{0i}) = \mu_n(\sigma_{i-2}...\sigma_0)^{-1}\Phi^{i-1,i,i+1...n} \cdot e^{t_{i-1,i}} \cdot (\Phi^{i-1,i,i+1...n})^{-1}\mu_n(\sigma_{i-2}...\sigma_0)$. There exists $y_i \in \hat{\mathfrak{t}}_{n+1}$ such that $\mu_n(\sigma_{i-2}...\sigma_0) = e^{y_i} s_{i-2}...s_0$, so for some $\tilde{y}_i \in \hat{\mathfrak{t}}_{n+1}$,

$$(\Phi^{i-1,i,i+1...n})^{-1}\mu_n(\sigma_{i-2}...\sigma_0) = s_{i-2}...s_0 e^{\tilde{y}_i}.$$

Then $\mu_n(x_{0i}) = e^{-\tilde{y}_i}(s_{i-2}...s_0)^{-1}e^{t_{i-1,i}}s_{i-2}...s_0 e^{\tilde{y}_i} = e^{-\tilde{y}_i}e^{t_{0i}}e^{\tilde{y}_i}$. As the action of $\hat{\mathfrak{t}}_{n+1}$ on $\hat{\mathfrak{f}}_n$ is by tangential automorphisms, we have $e^{-\tilde{y}_i}e^{t_{0i}}e^{\tilde{y}_i} = e^{z_i}e^{t_{0i}}e^{-z_i}$ for some $z_i \in \hat{\mathfrak{f}}_n$. So $\mu_n \circ (e^{t_{0i}} \mapsto x_{0i}) \in \mathrm{TAut}_n$. The general case follows from the identity $\mu_{O'}(b) = \Phi_{O,O'}\mu_O(b)\Phi_{O,O'}^{-1}$ and the fact that for any $\Psi \in \exp(\hat{\mathfrak{t}}_{n+1})$, $x \mapsto \Psi x \Psi^{-1}$ induces a tangential automorphism of $\exp(\hat{\mathfrak{f}}_n)$. □

**Proposition 3.2.** *If moreover* $O = \bullet \otimes \bar{O}$*, where* $\bar{O} \in \mathrm{Ob}(\mathbf{PaB})$ *has length* $n$*, then* $\mu_O(X_1...X_n) = e^{x_1+...+x_n}$.

*Proof.* The map $\mathrm{PB}_2 \to \mathrm{PB}_{n+1}$, $p \mapsto p^{0,\widetilde{1...n}}$ takes $x_{01}$ to $x_{01}...x_{0n} = X_1...X_n$. Similarly to (9), one proves that the diagram

$$\begin{array}{ccc} \mathrm{PB}_2 & \overset{p\mapsto p^{0,\widetilde{1...n}}}{\longrightarrow} & \mathrm{PB}_{n+1} \\ {\scriptstyle\tilde{\mu}_{\bullet\bullet}}\downarrow & & \downarrow{\scriptstyle\tilde{\mu}_{\bullet\otimes\bar{O}}} \\ \exp(\hat{\mathfrak{t}}_2) & \overset{x\mapsto x^{0,1...n}}{\longrightarrow} & \exp(\hat{\mathfrak{t}}_{n+1}) \end{array}$$

commutes. □

The various isomorphisms $\mu_O$ are related by the identities

$$\mu_{O'} = \mathrm{Ad}(\Phi_{O,O'}) \circ \mu_O; \tag{7}$$

the automorphisms $\mathrm{Ad}(\Phi_{O,O'})$ are no longer necessarily inner.

## 4. The identity $\mu_{O^{(i)}} = \mu_O^{1,2,...,ii+1,...,n} \circ \mu_{\bullet(\bullet\bullet)}^{i,i+1}$

Let $O \in \mathrm{Ob}(\mathbf{PaB})$ be a parenthesized word of length $n$; its letters are numbered $0, ..., n-1$. Let $i \in \{1, ..., n-1\}$, let $O^{(i)}$ be the object obtained by replacing the letter $\bullet$ numbered $i$ by $(\bullet\bullet)$ (e.g., if $O = \bullet(\bullet\bullet)$, then $O^{(1)} = \bullet((\bullet\bullet)\bullet))$. The purpose of this section is to show the identity

$$\mu_{O^{(i)}} = \mu_O^{1,2,...,ii+1,...,n} \circ \mu_{\bullet(\bullet\bullet)}^{i,i+1}.$$

4.1. **Free magmas and semigroups.** Recall that a magma is a triple $(M, M \times M \to M, e \in M)$ satisfying $e \times m \mapsto m$ and $m \times e \mapsto m$. A semigroup is a magma, where $M \times M \to M$ is associative.

Let $X$ be a finite set. Let $\mathrm{Mg}_X$ be the free magma generated by $X$ and $\mathrm{Sg}_X$ the semigroup generated by $X$.The assignments $X \mapsto \mathrm{Sg}_X$, $X \mapsto \mathrm{Mg}_X$ are functorial and we have a natural map $Mg_X \to \mathrm{Sg}_X$; so we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mg}_X & \to & \mathrm{Sg}_X \\ \downarrow & & \downarrow \\ \mathrm{Mg}_{\{\bullet\}} & \to & \mathrm{Sg}_{\{\bullet\}} = \mathbb{N} \end{array}$$

This diagram is Cartesian, so $\mathrm{Mg}_X$ can be identified with a fibered product. Explicitly, we have $\mathrm{Sg}_X = \sqcup_{n\geq 0} X^n$, $\mathrm{Mg}_{\{\bullet\}} = \sqcup_{n\geq 0}\{$parenthesizations of the word $\bullet ... \bullet$ of length $n\} = \sqcup_{n\geq 0}\{$rooted planar binary trees with $n$ leaves$\}$, $\mathrm{Mg}_X = \sqcup_{n\geq 0}\{$parenthesized words of length $n$ in the alphabet $X\}$.

We denote by $w : \mathrm{Mg}_X \to \mathrm{Sg}_X$ (word), $P : \mathrm{Mg}_X \to \mathrm{Mg}_{\{\bullet\}}$ (parenthesization) the natural maps; the various maps to $\mathbb{N}$ are denoted by $x \mapsto |x|$ (length).

Note that $S_n$ acts on $X^n$. For $w, w' \in \mathrm{Sg}_X$, with $|w| = |w'| = n$, we then set $S_{w,w'} = \{\sigma \in S_n | \sigma \cdot w = w'\}$.

4.2. **A braided monoidal category $\mathbf{PaB}_X$.** We denote by BMC the 'category' of braided monoidal categories (b.m.c.), where morphisms are the tensor functors.

We define a functor Sets $\to$ BMC, $X \mapsto \mathbf{PaB}_X$, adjoint to the 'objects' functor BMC $\to$ Sets, $\mathcal{C} \mapsto \mathrm{Ob}\,\mathcal{C}$. This means that for any set $X$ and b.m.c. $\mathcal{C}$, we have a natural bijection $\mathrm{Mor}_{\mathrm{Sets}}(X, \mathrm{Ob}\,\mathcal{C}) \simeq \mathrm{Mor}_{\mathrm{BMC}}(\mathbf{PaB}_X, \mathcal{C})$. More precisely, we have an injection $X \subset \mathrm{Ob}\,\mathbf{PaB}_X$, and for any b.m.c. $\mathcal{C}$ and any map $X \to \mathrm{Ob}\,\mathcal{C}$, there is attached a tensor functor $\mathbf{PaB}_X \to \mathcal{C}$, such that $\mathrm{Ob}\,\mathbf{PaB}_X \to \mathrm{Ob}\,\mathcal{C}$ extends $X \to \mathrm{Ob}\,\mathcal{C}$. When $X = \{\bullet\}$, $\mathbf{PaB}_X$ identifies with Bar-Natan's $\mathbf{PaB}$.

We now construct $\mathbf{PaB}_X$. We set $\mathrm{Ob}(\mathbf{PaB}_X) := \mathrm{Mg}_X$. For $O, O' \in \mathrm{Mg}_X$, we set $\mathbf{PaB}_X(O, O') = \emptyset$ if $|O| \neq |O'|$, and $= \mathrm{B}_n \times_\pi S_{w(O),w(O')}$ if $|O| = |O'| = n$ ($\pi : \mathrm{B}_n \to S_n$ is the canonical projection). So $\mathbf{PaB}_X(O, O') \subset \mathrm{B}_n$; since $S_{w,w'}S_{w',w''} \subset S_{w,w''}$, the product in $\mathrm{B}_n$ restricts to a map $\mathbf{PaB}_X(O, O') \times \mathbf{PaB}_X(O', O'') \to \mathbf{PaB}_X(O, O'')$, which we define as the composition in $\mathbf{PaB}_X$.

The tensor product is defined at the level of objects by the product in $\mathrm{Mg}_X$, and at the level of morphisms is induced by the juxtaposition map $\mathrm{B}_n \times \mathrm{B}_m \to \mathrm{B}_{n+m}$.

We now construct the braiding and associativity constraints. For $O, O', O'' \in Mg_X$, $a_{O,O',O''} \in \mathbf{PaB}_X((O \otimes O') \otimes O'', O \otimes (O' \otimes O''))$ is defined as the identity element in $\mathrm{B}_{n+n'+n''}$ ($n = |O|$, $n' = |O'|$, $n'' = |O''|$).

Then $\beta_{O,O'} \in \mathbf{PaB}_X(O \otimes O', O' \otimes O) \simeq \mathrm{B}_{|O|+|O'|}$ corresponds to $\sigma_{n,n'}$ (one checks that the image $s_{n,n'} \in S_{n+n'}$ of $\sigma_{n,n'}$ belongs to the desired $S_{w,w'}$).

One checks that $\mathbf{PaB}_X$, equipped with this structure, is a b.m.c., and that $X \mapsto \mathbf{PaB}_X$ is adjoint to the 'objects' functor.

4.3. **The category $\mathbf{PaCD}_X$.** We first define a tensor category $F_X$ as follows. $\mathrm{Ob}(F_X) := \mathrm{Sg}_X$, and for $w, w' \in \mathrm{Sg}_X$, $F_X(w, w') = \emptyset$ if $|w| \neq |w'|$, and $= (\exp(\hat{\mathfrak{t}}_n) \rtimes S_n) \times_\pi S_{w,w'}$ else, where $\pi : \exp(\hat{\mathfrak{t}}_n) \rtimes S_n \to S_n$ is the canonical projection. The composition is defined as above, using the product in $\exp(\hat{\mathfrak{t}}_n) \rtimes S_n$, again using $S_{w,w'}S_{w',w''} \subset S_{w,w''}$.

The tensor product is defined, at the level of objects, by the semigroup law, and at the level of morphisms using the juxtaposition $(\exp(\hat{\mathfrak{t}}_n) \rtimes S_n) \times (\exp(\hat{\mathfrak{t}}_{n'}) \rtimes S_{n'}) \to \exp(\hat{\mathfrak{t}}_{n+n'}) \rtimes S_{n+n'}$.

Let $\Phi \in M_1(\mathbf{k})$. For $X = \{\bullet\}$, $\mathrm{Sg}_X = \mathbb{N}$ (we then have $n \otimes m = n + m$). For $n, n', n'' \in \mathbb{N}$, we then set

$$a_{n,n',n''} := \Phi^{1...n,n+1...n+n',n+n'+1...n+n'+n''} \in \exp(\hat{\mathfrak{t}}_{n+n'+n''}) \in F_{\{\bullet\}}(n \otimes n' \otimes n'');$$

$s_{n,n'} \in S_{n+n'}$ is the block permutation $i \mapsto n' + i$ ($i \in [n]$), $n + i \mapsto i$ ($i \in [n']$) and

$$\beta_{n,n'} := (e^{t_{12}/2})^{1...n,n+1...n+n'} s_{n,n'} \in \exp(\hat{\mathfrak{t}}_{n+n'}) \rtimes S_{n+n'} = F_{\{\bullet\}}(n \otimes n', n' \otimes n).$$

We note that if $X$ is arbitrary and $w, w', w'' \in \mathrm{Sg}_X$, then $a_{|w|,|w'|,|w''|} \in F_X(w \otimes w' \otimes w'')$ and $\beta_{|w|,|w'|} \in F_X(w \otimes w', w' \otimes w)$.

We define the category $\mathbf{PaCD}_X$ by $\mathrm{Ob}(\mathbf{PaCD}_X) := \mathrm{Mg}_X$, and for $O, O' \in \mathrm{Mg}_X$, we set $\mathbf{PaCD}_X(O,O') := F_X(w(O), w(O'))$. The tensor product is defined at the level of objects as the product in $\mathrm{Mg}_2$; as $w : \mathrm{Mg}_2 \to \mathrm{Sg}_2$ is compatible with products, a tensor product is defined at the level of morphisms by $\mathbf{PaCD}_X(O_1,O_2) \otimes \mathbf{PaCD}_X(O_1',O_2') = F_X(w(O_1),w(O_2)) \otimes F_X(w(O_1'),w(O_2')) \to F_X(w(O_1)\otimes w(O_1'), w(O_2)\otimes w(O_2')) = F_X(w(O_1\otimes O_1'), w(O_2\otimes O_2')) = \mathbf{PaCD}_X(O_1\otimes O_1', O_2\otimes O_2')$.

Let $\Phi \in M_1(\mathbf{k})$. Then $\Phi$ gives rise to a b.m.c. structure on $\mathbf{PaCD}_X$ by $a_{O,O',O''} := a_{|O|,|O'|,|O''|} \in F_X(w(O)\otimes w(O')\otimes w(O'')) = \mathbf{PaCD}_X((O\otimes O')\otimes O'', O\otimes(O'\otimes O''))$ and $\beta_{O,O'} := \beta_{|O|,|O'|} \in F_X(w(O)\otimes w(O'), w(O')\otimes w(O)) = \mathbf{PaCD}_X(O\otimes O', O'\otimes O)$ for $O, O', O'' \in \mathrm{Mg}_X$.

We denote by $\mathbf{PaCD}_X^\Phi$ the resulting b.m.c.

4.4. **Tensor functors.** When $X = X_1 := \{\bullet\}$, $\mathbf{PaB}_X$ coincides with $\mathbf{PaB}$; we also denote $\mathrm{Mg}_X$, $\mathbf{PaCD}_X^\Phi$ by $\mathrm{Mg}$, $\mathbf{PaCD}^\Phi$. For $X = X_2 := \{\bullet, \circ\}$, we denote $\mathbf{PaB}_X$, $\mathbf{PaCD}_X^\Phi$, $\mathrm{Mg}_X$, $\mathrm{Sg}_X$ by $\mathbf{PaB}_2$, $\mathbf{PaCD}_2^\Phi$, $\mathrm{Mg}_2$, $\mathrm{Sg}_2$.

We define $\mathbf{PaB}_2 \to \mathbf{PaB}$ as the tensor functor induced by the map $X_2 \to \mathrm{Mg}_1$, $\bullet \mapsto \bullet$, $\circ \mapsto \bullet\bullet$.

We denote by $\mathbf{PaB} \to \mathbf{PaCD}^\Phi$ the tensor functor induced by the canonical injection $X_1 \to \mathrm{Ob}(\mathbf{PaCD}^\Phi) = \mathrm{Mg}_1$.

Similarly, we denote by $\mathbf{PaB}_2 \to \mathbf{PaCD}_2^\Phi$ the tensor functor induced by the canonical injection $X_2 \to \mathrm{Ob}(\mathbf{PaCD}_2^\Phi) = \mathrm{Mg}_2$.

Let us now construct a functor $F_{X_2} \to F_{X_1}$. At the level of objects, this is the semigroup morphism $\mathrm{Sg}_2 \to \mathrm{Sg}_1$ induced by the map $l : X_2 \to \mathrm{Sg}_1 \simeq \mathbb{N}$, $w \mapsto \tilde w$ given by $\bullet \mapsto 1$ and $\circ \mapsto 2$. So for $w = (w_1, ..., w_n) \in \sqcup_{n\geq 0} X_2^n$, $\tilde w = \sum_{i=1}^n l(w_i)$, where $l(\bullet) = 1$ and $l(\bullet\bullet) = 2$. Let us now define the functor at the level of morphisms, i.e. the maps $F_{X_2}(w,w') \to F_{X_1}(\tilde w, \tilde w')$. As $F_{X_2}(w,w') = \emptyset$ unless $(\mathrm{card}\{i|w_i = \bullet\}, \mathrm{card}\{i|w_i = \circ\}) = (\mathrm{card}\{i|w_i' = \bullet\}, \mathrm{card}\{i|w_i' = \circ\})$, we will assume that these pairs of integers are equal (in particular $|w| = |w'|$); we denote this pair by $(n_1, n_2)$. Note that $|w| = |w'| = n_1 + n_2$, while $\tilde w = \tilde w' = n_1 + 2n_2$.

There is a unique non-decreasing map $\phi_w : [n_1 + 2n_2] \to [n_1 + n_2]$, such that $i$ has one preimage by $\phi_w$ if $w_i = \bullet$ and two preimages if $w_i = \circ$; for example, if $w = (\bullet, \bullet, \circ, \circ, \bullet)$, then $\phi_w : [7] \to [5]$ is $(1, ..., 7) \mapsto (1,2,3,3,4,4,5)$.

Moreover, for any $\sigma \in S_{n_1+n_2}$, there is a unique $\sigma^w \in S_{n_1+2n_2}$ such that: (a) $\sigma\circ\phi_w = \phi_{w'}\circ\sigma^w$, where $w' = \sigma\cdot w$, so that $\sigma^w$ restricts to bijections $\phi_w^{-1}(i) \to \phi_{w'}^{-1}(i)$; (b) these bijections are increasing (this condition is nonempty only if $\mathrm{card}\,\phi_w^{-1}(i) > 1$). The map $\sigma \mapsto \sigma'$ is a group morphism $S_{n_1+n_2} \to S_{n_1+2n_2}$ (it maps a permutation to a block permutation); for example, if $w = (\circ, \bullet, \bullet)$, this map is $S_3 \to S_4$, $\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}1&2&3&4\\3&1&2&4\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}1&2&3&4\\1&2&4&3\end{smallmatrix}\right)$.

The morphisms $\mathfrak{t}_{n_1+n_2} \to \mathfrak{t}_{n_1+2n_2}$, $x \mapsto x^{\phi_w}$ and $S_{n_1+n_2} \to S_{n_1+2n_2}$, $\sigma \mapsto \sigma^w$ are compatible, so we obtain a group morphism $\exp(\hat{\mathfrak{t}}_{n_1+n_2}) \rtimes S_{n_1+n_2} \to \exp(\hat{\mathfrak{t}}_{n_1+2n_2}) \rtimes S_{n_1+2n_2}$. We then define $F_{X_2}(w,w') \to F_{X_1}(\tilde w, \tilde w')$ as the restriction of this group morphism. One checks that this map is compatible with tensor products, so we have defined a tensor functor $F_{X_2} \to F_{X_1}$.

The tensor functor $F_{X_2} \to F_{X_1}$ extends to a tensor functor $\mathbf{PaCD}_2^\Phi \to \mathbf{PaCD}^\Phi$ as follows. There is a unique magma morphism $\mathrm{Mg}_2 \to \mathrm{Mg}_1$, $O \mapsto \tilde O$, extending the map $X_2 \to \mathrm{Mg}_1$, $\bullet \mapsto \bullet$, $\circ \mapsto \bullet\bullet$. It is such that the diagram

$$\begin{array}{ccc} \mathrm{Mg}_2 & \to & \mathrm{Mg}_1 \\ \downarrow & & \downarrow \\ \mathrm{Sg}_2 & \to & \mathrm{Sg}_1 \end{array}$$

commutes. The functor $\mathbf{PaCD}_2^\Phi \to \mathbf{PaCD}^\Phi$ is defined, at the level of objects, as the map $\mathrm{Mg}_2 \to \mathrm{Mg}_1$ and at the level of morphisms by $\mathbf{PaCD}_2^\Phi(O,O') = F_{X_2}(w(O), w(O')) \to F_{X_1}(\widetilde{w(O)}, \widetilde{w(O')}) = F_{X_1}(w(\tilde O), w(\tilde O')) = \mathbf{PaCD}(O,O')$.

It remains to show that it takes braidings and associativity constraints to their analogues. Namely:

(a) it takes $\beta_{O,O'} \in \mathbf{PaCD}_2(O \otimes O', O' \otimes O)$ to $\beta_{\tilde{O},\tilde{O}'} \in \mathbf{PaCD}(\tilde{O} \otimes \tilde{O}', \tilde{O}' \otimes \tilde{O})$.

(b) it takes $a_{O,O',O''} \in \mathbf{PaCD}_2((O \otimes O') \otimes O'', O \otimes (O' \otimes O''))$ to $a_{\tilde{O},\tilde{O}',\tilde{O}''} \in \mathbf{PaCD}((\tilde{O} \otimes \tilde{O}') \otimes \tilde{O}'', \tilde{O} \otimes (\tilde{O}' \otimes \tilde{O}''))$.

To prove (a), let $w, w' = w(O), w(O')$, $(\mathrm{card}\{i|w_i = \bullet\}, \mathrm{card}\{i|w_i = \circ\}) = (n_1, n_2)$, $(\mathrm{card}\{i|w'_i = \bullet\}, \mathrm{card}\{i|w'_i = \circ\}) = (n'_1, n'_2)$. Then $\beta_{O,O'} = \beta_{n_1+n_2,n'_1+n'_2} \in F_{X_2}(w \otimes w', w' \otimes w)$. Similarly, $\beta_{\tilde{O},\tilde{O}'} = \beta_{n_1+2n_2,n'_1+2n'_2} \in F_{X_1}(\tilde{w} \otimes \tilde{w}', \tilde{w}' \otimes \tilde{w})$.

Now note that:

$$((t_{12})^{1\ldots n,n+1\ldots n+n'})^{\phi_{w\otimes w'}} = (t_{12})^{1\ldots n_1+2n_2,n_1+2n_2+1\ldots n_1+2n_2+n'_1+2n'_2},$$

and

$$(s_{n,n'})^{w\otimes w'} = s_{n_1+2n_2,n'_1+2n'_2}.$$

So the map $F_{X_2}(w \otimes w', w' \otimes w) \to F_{X_1}(\tilde{w} \otimes \tilde{w}', \tilde{w}' \otimes \tilde{w})$ takes $\beta_{n,n'}$ to $\beta_{n_1+2n_2,n'_1+2n'_2}$. The proof of (b) is similar.

Then the diagram of functors

$$\begin{array}{ccc} \mathbf{PaB}_2 & \to & \mathbf{PaB} \\ \downarrow & & \downarrow \\ \mathbf{PaCD}_2^{\Phi} & \to & \mathbf{PaCD}^{\Phi} \end{array}$$

commutes by universal properties (the two composed functors $\mathbf{PaB}_2 \to \mathbf{PaCD}^{\Phi}$ coincides as their restrictions to the elements of $X_2 \subset \mathrm{Ob}(\mathbf{PaB}_2)$ do).

**Remark 4.1.** More generally, to any map $X \to \mathrm{Mg}_1$, one associates a tensor functor $\mathbf{PaCD}_X^{\Phi} \to \mathbf{PaCD}^{\Phi}$, defined at the level of objects by the extension of this map to a morphism $\mathrm{Mg}_X \to \mathrm{Mg}_1$ and at the level of morphisms by suitable iterations of cobrackets, and it is such that

$$\begin{array}{ccc} \mathbf{PaB}_X & \to & \mathbf{PaB} \\ \downarrow & & \downarrow \\ \mathbf{PaCD}_X & \to & \mathbf{PaCD} \end{array}$$

commutes.

4.5. **Relation between braid groups representations.** Let $n \geq 1$, let $i \in [n]$, let $w^i = (\bullet, \ldots, \bullet, \circ, \bullet, \ldots, \bullet) \in \mathrm{Sg}_2$ be given by $w_i = \circ$ and $w_j = \bullet$ for $j \in [n] - \{i\}$. Let $O \in \mathrm{Mg}_2$ be such that $w(O) = w^i$. We have proved that the diagram

$$\begin{array}{ccc} \mathbf{PaB}_2(O) & \to & \mathbf{PaB}(\tilde{O}) \\ \downarrow & & \downarrow \\ \mathbf{PaCD}_2(O) & \to & \mathbf{PaCD}(\tilde{O}) \end{array}$$

commutes.

We have isomorphisms:

$\mathbf{PaB}_2(O) \simeq \mathrm{B}_n \times_\pi S_{n-1}$, where $S_{n-1} \subset S_n$ identifies with $\{\sigma \in S_n | \sigma(i) = i\}$;

$\mathbf{PaCD}_2(O) \simeq (\exp(\hat{\mathfrak{t}}_n) \rtimes S_n) \times_\pi S_{n-1}$;

$\mathbf{PaB}(\tilde{O}) \simeq \mathrm{B}_{n+1}$;

$\mathbf{PaCD}(\tilde{O}) \simeq \exp(\hat{\mathfrak{t}}_{n+1}) \rtimes S_{n+1}$.

For $O \in \mathrm{Mg}_{X_1}$, $|O| = n$, the morphism $\mathbf{PaB}(O) \to \mathbf{PaCD}(O)$ is a morphism $\mu_O : \mathrm{B}_n \to \exp(\hat{\mathfrak{t}}_n) \rtimes S_n$. Note that if $O_X \in \mathrm{Mg}_X$ and $O := P(O_X)$, then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{PaB}_X(O_X) & \to & \mathbf{PaCD}_X(O_X) \\ \downarrow & & \downarrow \\ \mathrm{B}_n & \stackrel{\mu_O}{\to} & \exp(\hat{\mathfrak{t}}_n) \rtimes S_n \end{array}$$

where the vertical maps are injective.

The above commutative diagram therefore inserts in a diagram

$$(8)\qquad \begin{array}{ccccc} \mathrm{B}_n & \leftarrow & \mathrm{B}_n \times_\pi S_{n-1} & \overset{1,2,\ldots,\widetilde{ii+1},\ldots,n}{\to} & \mathrm{B}_{n+1} \\ {\scriptstyle \mu_O}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mu_{O^{(i)}}} \\ \exp(\hat{\mathfrak{t}}_n) \rtimes S_n & \leftarrow & (\exp(\hat{\mathfrak{t}}_n) \rtimes S_n) \times_\pi S_{n-1} & \overset{1,2,\ldots,ii+1,\ldots,n}{\to} & \exp(\hat{\mathfrak{t}}_{n+1}) \rtimes S_{n+1} \end{array}$$

Restricting to pure braid groups, we obtain the commutative diagram

$$(9)\qquad \begin{array}{ccc} \mathrm{PB}_n & \overset{1,2,\ldots,\widetilde{ii+1},\ldots,n}{\to} & \mathrm{PB}_{n+1} \\ {\scriptstyle \mu_O}\downarrow & & \downarrow{\scriptstyle \mu_{O^{(i)}}} \\ \exp(\hat{\mathfrak{t}}_n) & \overset{1,2,\ldots,ii+1,i+2,\ldots,n+1}{\to} & \exp(\hat{\mathfrak{t}}_{n+1}) \end{array}$$

4.6. **Relation between $\mu_O$ and $\mu_{O^{(i)}}$.** Let $O \in \mathrm{Ob}(\mathbf{PaB})$, $|O| = n$. We index letters in $O$ by $0, \ldots, n-1$, fix an index $i \neq 0$ and construct $O^{(i)}$ by doubling inside $O$ the $\bullet$ with index $i$.

$O$ gives rise to a morphism $\tilde{\mu}_O : \mathrm{B}_n \to \exp(\hat{\mathfrak{t}}_n) \rtimes S_n$, which restricts to $\mu_O : \mathrm{F}_{n-1} \to \exp(\hat{\mathfrak{f}}_{n-1})$. Similarly, $\tilde{\mu}_{O^{(i)}} : \mathrm{B}_{n+1} \to \exp(\hat{\mathfrak{t}}_{n+1}) \rtimes S_{n+1}$ restricts to $\mu_{O^{(i)}} : \mathrm{F}_n \to \exp(\hat{\mathfrak{f}}_n)$.

We want to prove that

$$(10)\qquad \mu_{O^{(i)}} = \mu_O^{1,2,\ldots,ii+1,\ldots,n} \circ \mu_{\bullet(\bullet\bullet)}^{i,i+1}.$$

We first show that there are uniquely determined elements $g_1, \ldots, g_{n-1} \in \exp(\hat{\mathfrak{f}}_{n-1})$ and $g, h \in \exp(\hat{\mathfrak{f}}_2)$ such that:

(a) $\mu_O = [\![g_1(x_1, \ldots, x_{n-1}), \ldots, g_{n-1}(x_1, \ldots, x_{n-1})]\!]$, $\log g_i(x_1, \ldots, x_{n-1}) = -\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(x^2)$, and[8]

(b) $\mu_{\bullet(\bullet\bullet)} = [\![g(x_1, x_2), h(x_1, x_2)]\!]$, $\log g(x_1, x_2) = O(x^2)$, $\log h(x_1, x_2) = -\frac{1}{2}x_1 + O(x^2)$.

Let us prove the first statement (it actually contains the second statement as a particular case). The elements $g_i(x_1, \ldots, x_{n-1})$ are uniquely determined by the equality $\mu_O = [\![g_1, \ldots, g_{n-1}]\!]$, together with the condition that the coefficient of $x_i$ in the expansion of $\log g_i$ vanishes. We should then prove that $\log g_i = -\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(x^2)$. We have

$$\tilde{\mu}_O(\sigma_j) = e^{a_j} \cdot e^{t_{j-1,j}/2} s_j \cdot e^{-a_j},$$

where $a_j \in \hat{\mathfrak{t}}_n$ has valuation $\geq 2$ (we write this as $a_j \in O(t^2)$), and

$$\mu_O(X_i) = \tilde{\mu}_O(\sigma_1)^{-1} \ldots \tilde{\mu}_O(\sigma_{i-1})^{-1} \tilde{\mu}_O(\sigma_i)^2 \tilde{\mu}_O(\sigma_{i-1}) \ldots \tilde{\mu}_O(\sigma_1).$$

Now

$$\tilde{\mu}_O(\sigma_{i-1}) \ldots \tilde{\mu}_O(\sigma_1) = s_{i-1} \ldots s_1 e^{\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(t^2)}$$

and $\tilde{\mu}_O(\sigma_i^2) = e^{a_i} e^{t_{i-1,i}} e^{-a_i}$. It follows that

$$\mu_O(X_i) = e^{-\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(t^2)} e^{\tilde{a}_i} \cdot e^{x_i} \cdot (same)^{-1},$$

where $\tilde{a}_i = s_1 \ldots s_{i-1} \cdot a_i \cdot s_{i-1} \ldots s_1 \in O(t^2)$, so $\mu_O(X_i) = e^{-\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(t^2)} \cdot e^{x_i} \cdot (same)^{-1}$, which implies that $g_i$ has the announced form.

To prove (10), we need to prove the equality

$$(11)\qquad \begin{aligned} \mu_{O^{(i)}} = [\![ & g_1(x_1, \ldots, x_i + x_{i+1}, \ldots, x_n), \ldots, g_i(x_1, \ldots, x_i + x_{i+1}, \ldots, x_n) g(x_i, x_{i+1}), \\ & g_i(x_1, \ldots, x_i + x_{i+1}, \ldots, x_n) h(x_i, x_{i+1}), \ldots, g_{n-1}(x_1, \ldots, x_i + x_{i+1}, \ldots, x_n)]\!]. \end{aligned}$$

(9) implies that the diagram

$$\begin{array}{ccc} \mathrm{F}_{n-1} & \to & \mathrm{F}_n \\ {\scriptstyle \mu_O}\downarrow & & \downarrow{\scriptstyle \mu_{O^{(i)}}} \\ \exp(\hat{\mathfrak{f}}_{n-1}) & \to & \exp(\hat{\mathfrak{f}}_n) \end{array}$$

[8] $O(x^2)$ means an element of $\hat{\mathfrak{f}}_{n-1}$ of valuation $\geq 2$.

commutes, where the upper morphism takes $X_j$ ($j \in [n-1]$) to: $X_j$ if $j < i$, $X_i X_{i+1}$ if $j = i$, $X_{j+1}$ if $j > i+1$ and the lower morphism is similarly defined (replacing products by sums and $X_k$'s by $x_k$'s). Specializing to the generators $X_j$ ($j \neq i$) of $\mathrm{F}_{n-1}$, this yields

$$\mu_{O^{(i)}}(X_j) = g_j^{0,1,\ldots,ii+1,\ldots,n} \cdot e^{x_j} \cdot (same)^{-1}$$

for $j < i$ and

$$\mu_{O^{(i)}}(X_j) = g_{j-1}^{0,1,\ldots,ii+1,\ldots,n} \cdot e^{x_j} \cdot (same)^{-1}$$

for $j > i+1$, which implies that (11) holds when applied to the generators $X_j$, $j \neq i, i+1$.

We now prove that (11) also holds when applied to $X_i$ and $X_{i+1}$.

The morphism $X_i \in \mathrm{B}_n = \mathbf{PaB}(O, O)$ can be decomposed as

$$O \overset{(\sigma_{i-2}\ldots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-2}\ldots\sigma_0}{\to} O.$$

Here the braid group elements indicate the morphisms. Let $\gamma \in \exp(\hat{\mathfrak{t}}_n) \rtimes S_n$ be the image of the morphism $O \overset{(\sigma_{i-2}\ldots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ under $\mathbf{PaB} \to \mathbf{PaCD}$; its image in $S_n$ is the permutation $s_0 \ldots s_{i-2}$, i.e., $(0, \ldots, n-1) \mapsto (i-1, 0, 1, \ldots, i-2, i, i+1, \ldots, n-1)$. The image of $(O_1 \otimes (\bullet\bullet)) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ is $e^{t_{i-1,i}}$, therefore the image of $X_i$ is

$$\mu_O(X_i) = \gamma e^{t_{i-1,i}} \gamma^{-1}.$$

We have $\gamma = \gamma_0 s_0 \ldots s_{i-2}$, where $\gamma_0 \in \exp(\hat{\mathfrak{t}}_n)$. As $s_0 \ldots s_{i-2} \cdot t_{i-1,i} = x_i$, we have

$$\mu_O(X_i) = \gamma_0 e^{x_i} \gamma_0^{-1}.$$

As this image is also $g_i(x_1, \ldots, x_{n-1}) \cdot e^{x_i} \cdot (same)^{-1}$, we derive from this that $g_i^{-1}\gamma_0$ commutes with $x_i$, hence by Proposition A.1 has the form $e^{\lambda x_i} \alpha^{0i,1,2,\ldots,i-1,i+1,\ldots,n-1}$, where $\alpha \in \exp(\hat{\mathfrak{t}}_{n-1})$.

Since $\mu_O(\sigma_j) = s_j e^{t_{j,j+1}/2}$, we get $\log \gamma_0 = -\frac{1}{2}(x_1 + \ldots + x_{i-1}) + O(x^2)$. Comparing linear terms in $x_i$, we get $\lambda = 0$.

Let us now compute $\mu_{O^{(i)}}(X_i)$. The morphism $X_i \in \mathrm{B}_{n+1} = \mathbf{PaB}(O^{(i)}, O^{(i)})$ can be decomposed as

$$O^{(i)} \overset{(\sigma_{i-2}\ldots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2 \overset{\sigma_{i-2}\ldots\sigma_0}{\to} O^{(i)}$$

(here $\sigma_{i-1}^2$ involves the two first $\bullet$ of $\bullet(\bullet\bullet)$). The morphism $O^{(i)} \overset{(\sigma_{i-2}\ldots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2$ is obtained from $O^{(i)} \overset{(\sigma_{i-2}\ldots\sigma_0)^{-1}}{\to} (O_1 \otimes (\bullet\bullet)) \otimes O_2$ by the operation of doubling of the $i$th strand, so its image is $\gamma^{0,1,2,\ldots,ii+1,\ldots,n} = \gamma_0^{0,1,2,\ldots,ii+1,\ldots,n}(s_0 \ldots s_{i-2})$. The image of $\bullet(\bullet\bullet) \overset{\sigma_1^2}{\to} \bullet(\bullet\bullet)$ is $g(x_1, x_2) \cdot e^{x_1} \cdot (same)^{-1}$, so the image of

$$(O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2 \overset{\sigma_{i-1}^2}{\to} (O_1 \otimes (\bullet(\bullet\bullet))) \otimes O_2$$

is $g(t_{i-1,i}, t_{i-1,i+1}) e^{t_{i-1,i}} (same)^{-1}$. It follows that

$$\mu_{O^{(i)}}(X_i) = \gamma^{0,1,2,\ldots,ii+1,\ldots,n} g(t_{i-1,i}, t_{i-1,i+1}) \cdot e^{t_{i-1,i}} \cdot (same)^{-1} = \gamma_0^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1}.$$

Now we claim that

$$\gamma_0^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) e^{x_i} (same)^{-1} = g_i^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1}.$$

Indeed,

$$\begin{aligned}
&(g_i^{-1}\gamma_0)^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1} \\
&= (\alpha^{0i,1,2,\ldots,i-1,i+1,\ldots,n-1})^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1} \\
&= \alpha^{0ii+1,2,3,\ldots,i-1,i+2,\ldots} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1}.
\end{aligned}$$

Now $x_i$ and $x_{i+1}$ commute with any $\alpha^{0ii+1,\cdots}$, so this is $g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1}$.

So we get

$$\mu_{O^{(i)}}(X_i) = g_i^{0,1,2,\ldots,ii+1,\ldots,n} g(x_i, x_{i+1}) \cdot e^{x_i} \cdot (same)^{-1}.$$

The same argument shows that

$$\mu_{O^{(i)}}(X_{i+1}) = g_i^{0,1,2,\ldots,ii+1,\ldots,n} h(x_i, x_{i+1}) \cdot e^{x_{i+1}} \cdot (same)^{-1},$$

as wanted.

## 5. The map $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$

We show that for $\Phi \in M_1(\mathbf{k})$, $\mu_\Phi \in \mathrm{SolKV}(\mathbf{k})$. By construction of $\mu_\Phi$, we have $\mu_\Phi(X) \sim e^x$, $\mu_\Phi(Y) \sim e^y$, so $\mu_\Phi \in \mathrm{TAut}_2$.

### 5.1. Proof of $\mathrm{Ad}\,\Phi(t_{12}, t_{23}) \circ \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}$. We first prove:

**Proposition 5.1.** *1)* $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$.
*2)* $\Phi_{\bullet((\bullet\bullet)\bullet),\bullet(\bullet(\bullet\bullet))} = \Phi(t_{12}, t_{23})$.

*Proof.* Let us prove 1). $x_{01} \in \mathrm{B}_3 = \mathbf{PaB}(\bullet(\bullet\bullet))$ corresponds to $a_{\bullet,\bullet,\bullet} \circ (\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_\bullet) \circ a^{-1}_{\bullet,\bullet,\bullet}$. The image of this element in $\exp(\hat{\mathfrak{t}}_3) \rtimes S_3$ is $\mu_{\bullet(\bullet\bullet)}(X) = \Phi(t_{01}, t_{12}) e^{t_{01}} \Phi(t_{01}, t_{12})^{-1}$. Since $t_{01}+t_{12}+t_{02}$ is central in $\mathfrak{t}_3$ and since $\Phi$ is group-like, this is $\Phi(t_{01}, -t_{01}-t_{02}) e^{t_{01}} \Phi(t_{01}, -t_{01}-t_{02})^{-1} = \Phi(x, -x-y) e^x \Phi(x, -x-y)^{-1} = \mu_\Phi(X)$. Similarly, $x_{02}$ corresponds to $(\mathrm{id}_\bullet \otimes \beta_{\bullet,\bullet}) \circ a_{\bullet,\bullet,\bullet} \circ (\beta^2_{\bullet,\bullet} \otimes \mathrm{id}_\bullet) \circ a^{-1}_{\bullet,\bullet,\bullet} \circ (\mathrm{id}_\bullet \otimes \beta^{-1}_{\bullet,\bullet})$. The image of this element in $\exp(\hat{\mathfrak{t}}_3) \rtimes S_3$ is

$$\begin{aligned}
&\mu_{\bullet(\bullet\bullet)}(Y) \\
&= e^{t_{12}/2}(12)\Phi(t_{01}, t_{12}) e^{t_{01}} \Phi(t_{01}, t_{12})^{-1}(12) e^{-t_{12}/2} = e^{t_{12}/2}\Phi(t_{02}, t_{12}) e^{t_{02}} \Phi(t_{02}, t_{12})^{-1} e^{-t_{12}/2} \\
&= e^{-(t_{01}+t_{02})/2}\Phi(t_{02}, -t_{01}-t_{02}) e^{t_{02}} \Phi(t_{02}, -t_{01}-t_{02})^{-1} e^{(t_{01}+t_{02})/2} \\
&= e^{-(x+y)/2}\Phi(y, -x-y) e^y \Phi(y, -x-y)^{-1} e^{(x+y)/2} = \mu_\Phi(Y).
\end{aligned}$$

So $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$.

Let us now prove 2). Let $O := \bullet((\bullet\bullet)\bullet)$, $O' := \bullet(\bullet(\bullet\bullet))$. Then $\mathrm{can}_{O,O'} = \mathrm{id}_\bullet \otimes a_{\bullet,\bullet,\bullet} \in \mathbf{PaB}(O, O')$, whose image in $\mathbf{PaCD}(O, O') = \exp(\hat{\mathfrak{t}}_4) \rtimes S_4$ is $\Phi(t_{12}, t_{23}) = \Phi_{O,O'}$. □

We now prove (2). Applying (4) to $O = \bullet(\bullet\bullet)$ and $i = 1, 2$, and using $\mu_{\bullet(\bullet\bullet)} = \mu_\Phi$, we get

$$\mu_{\bullet((\bullet\bullet)\bullet)} = \mu_\Phi^{12,3} \circ \mu_\Phi^{1,2}, \quad \mu_{\bullet(\bullet(\bullet\bullet))} = \mu_\Phi^{1,23} \circ \mu_\Phi^{2,3}.$$

Moreover, (7) implies

$$\mathrm{Ad}\,\Phi_{\bullet((\bullet\bullet)\bullet),\bullet(\bullet(\bullet\bullet))} \circ \mu_{\bullet((\bullet\bullet)\bullet)} = \mu_{\bullet(\bullet(\bullet\bullet))}.$$

As $\Phi_{\bullet((\bullet\bullet)\bullet),\bullet(\bullet(\bullet\bullet))} = \Phi(t_{12}, t_{23})$, we get (2).

### 5.2. Proof of $\mu_\Phi(XY) = e^{x+y}$. We will give three proofs:

*First proof.* We have

$$\begin{aligned}
&\mu_\Phi(XY) = \mu_\Phi(X)\mu_\Phi(Y) \\
&= \Phi(x, -x-y) e^x \Phi(-x-y, x) e^{-(x+y)/2} \Phi(y, -x-y) e^y \Phi(-x-y, x) e^{(x+y)/2} \\
&= \Phi(x, -x-y) e^{x/2} \Phi(y, x) e^{y/2} \Phi(-x-y, x) e^{(x+y)/2} = e^{x+y},
\end{aligned}$$

where the second equality follows from the duality identity and the third and fourth equalities both follow from the hexagon identity.

*Second proof.* Let us set $\nu := \mu_\Phi^{-1}$. Since $\mu_\Phi$ satisfies (2), we have

$$\nu^{2,3} \circ \nu^{1,23} = \nu^{1,2} \circ \nu^{12,3} \circ \mathrm{Ad}(\Phi(t_{12}, t_{23})). \tag{12}$$

Let us set $C(x, y) := \nu(x + y)$, and apply (12) to $x + y + z$ to obtain $C(x, C(y, z)) = C(C(x, y), z)$. According to [AT], this implies $C(x, y) = s^{-1} \log(e^{sx} e^{sy})$ for some $s \in \mathbf{k}^\times$. Checking degree 1 and 2 terms in $\nu$, we get $s = 1$.

*Third proof.* As $\tilde\mu_{\bullet\bullet}(x_{01}) = e^{t_{01}}$, and using Proposition 3.2, we get $\mu_{\bullet\otimes\bar O}(X_1...X_n) = \tilde\mu_{\bullet\otimes\bar O}(X_1...X_n) = (e^{t_{01}})^{0,1...n} = e^{x_1+...+x_n}$. This implies $\mu_\Phi(XY) = e^{x+y}$ since $\mu_\Phi = \mu_{\bullet(\bullet\bullet)}$.

5.3. **Proof that $J(\mu_\Phi)$ is a $\delta$-coboundary (end of proof of Theorem 2.1).** Since $J(\mathrm{Ad}\,\Phi(t_{12},t_{23})) = 0$, and $J(\mu_\Phi^{12,3}) = J(\mu_\Phi)^{12,3}$, etc., we get by applying $J$ to (2),

$$\Phi(t_{12},t_{23})\cdot J(\mu_\Phi)^{12,3} + \Phi(t_{12},t_{23})\circ\mu_\Phi^{12,3}\cdot J(\mu_\Phi)^{1,2} = J(\mu_\Phi)^{1,23} + \mu_\Phi^{1,23}\cdot J(\mu_\Phi)^{2,3}.$$

Applying the inverse of (2), we get

$$(\mu_\Phi^{1,2})^{-1}\circ(\mu_\Phi^{12,3})^{-1}\cdot J(\mu_\Phi)^{12,3} + (\mu_\Phi^{1,2})^{-1}\cdot J(\mu_\Phi)^{1,2} = (\mu_\Phi^{2,3})^{-1}\circ(\mu_\Phi^{1,23})^{-1}\cdot J(\mu_\Phi)^{1,23} + (\mu_\Phi^{2,3})^{-1}\cdot J(\mu_\Phi)^{2,3},$$

and since $a^{12,3}\cdot t^{12,3} = (a\cdot t)^{12,3}$, etc.,

$$(\mu_\Phi^{1,2})^{-1}\cdot(\mu_\Phi^{-1}\cdot J(\mu_\Phi))^{12,3} + (\mu_\Phi^{-1}\cdot J(\mu_\Phi))^{1,2} = (\mu_\Phi^{2,3})^{-1}\cdot(\mu_\Phi^{-1}\cdot J(\mu_\Phi))^{1,23} + (\mu_\Phi^{-1}\cdot J(\mu_\Phi))^{2,3}.$$

Now $\mu_\Phi^{-1}(x+y) = \log(e^x e^y)$ implies that $(\mu_\Phi^{1,2})^{-1}\cdot t^{12,3} = t^{\tilde{12},3}$, and similarly with $1,23$, so $\tilde\delta(\mu_\Phi^{-1}\cdot J(\mu_\Phi)) = 0$. So there exists $\gamma\in\hat{\mathfrak{T}}_1$ with valuation $\geq 2$ such that $\mu_\Phi^{-1}\cdot J(\mu_\Phi) = \tilde\delta(\gamma)$. Now $\mu_\Phi\cdot\gamma^{\tilde{12}} = \gamma^{12}$, and $\mu_\Phi\cdot\gamma^1 = \gamma^1$, $\mu_\Phi\cdot\gamma^2 = \gamma^2$ as $\mu_\Phi(x)\sim x$, $\mu_\Phi(y)\sim y$, therefore $\mu_\Phi\cdot\tilde\delta(\gamma) = \delta(\gamma)$. So $J(\mu_\Phi) = \delta(\gamma)$. It follows that for a suitable $\gamma\in u^2\mathbf{k}[[u]]$, we have $J(\mu_\Phi) = \delta(\gamma) = \langle\gamma(x+y)-\gamma(x)-\gamma(y)\rangle$.

All this ends the construction of the map $M_1(\mathbf{k})\to\mathrm{SolKV}(\mathbf{k})$, hence the proof of Theorem 2.1.

5.4. **Computation of $J(\mu_\Phi)$ (proof of Proposition 2.2).** Let $U := [\![1, A(x,y)]\!]\in\mathrm{TAut}_2$, where

$$\log A(x,y) = \sum_{k\geq 1}\alpha_k(\mathrm{ad}\,x)^k(y) + O(y^2)$$

(here $O(y^2)$ means a series of elements with $y$-degree $\geq 2$). Then $\log U = [\![0,\sum_{k\geq1}\alpha_k(\mathrm{ad}\,x)^k(y) + O(y^2)]\!]$, and $J(U) = j(\log U) + O(y^2)$. Now $j(\log U) = \langle\sum_{k\geq1}\alpha_k y(-x)^k + O(y^2)\rangle$. So

$$J(U) = \langle\sum_{k\geq1}\alpha_k(-x)^k y\rangle + O(y^2).$$

On the other hand, the hexagon identity implies that $\mu_\Phi = \mathrm{Inn}(\Phi(x,-x-y)e^{-x/2})\circ\bar\mu_\Phi$, where $\bar\mu_\Phi = [\![1,\Phi(x,y)^{-1}]\!]$, and we then have $J(\bar\mu_\Phi) = J(\mu_\Phi)$.

We have $\log\Phi(x,y) = -\sum_{k\geq1}\zeta_\Phi(k+1)(\mathrm{ad}\,x)^k(y) + O(y^2)$, therefore

$$J(\mu_\Phi) = J(\bar\mu_\Phi) = \langle\sum_{k\geq1}(-1)^k\zeta_\Phi(k+1)x^k y\rangle + O(y^2).$$

As we have $J(\mu_\Phi) = \langle f(x)+f(y)-f(x+y)\rangle$ for some series $f(x)$, we get

(13)
$$J(\mu_\Phi) = \langle(-1)^k\frac{\zeta_\Phi(k+1)}{k+1}((x+y)^{k+1}-x^{k+1}-y^{k+1})\rangle = \langle\log\Gamma_\Phi(x)+\log\Gamma_\Phi(y)-\log\Gamma_\Phi(x+y)\rangle.$$

This proves Proposition 2.2.

## 6. Group and torsor aspects

6.1. **Group structures of $\mathrm{KV}(\mathbf{k})$ and $\mathrm{KRV}(\mathbf{k})$.** It is proved in [AT] that $\mathrm{KRV}(\mathbf{k})$ is a group, acting freely and transitively on $\mathrm{SolKV}(\mathbf{k})$.

Let us prove that $\mathrm{KV}(\mathbf{k})$ is a group. For $\alpha\in\mathrm{KV}(\mathbf{k})$, let $\sigma_\alpha := \mathrm{Duf}(\alpha)$, so $\sigma_\alpha\in u^2\mathbf{k}[[u]]$, and $J(\alpha) = \tilde\delta(\sigma_\alpha)$. If $\alpha,\alpha'\in\mathrm{KV}(\mathbf{k})$, we have clearly $\alpha'\circ\alpha(X)\sim X$, $\alpha'\circ\alpha(Y)\sim Y$, $\alpha'\circ\alpha(XY) = XY$. Moreover, $J(\alpha'\circ\alpha) = J(\alpha')+\alpha'\cdot J(\alpha) = \tilde\delta(\sigma_{\alpha'})+\alpha'\cdot\tilde\delta(\sigma_\alpha) = \tilde\delta(\sigma_\alpha+\sigma_{\alpha'})$, where the last equality follows from $\alpha'(X)\sim X$, $\alpha'(Y)\sim Y$, $\alpha'(XY) = XY$, which implies $\tilde\delta(\alpha'\cdot t) = \tilde\delta(t)$ for $t\in\hat{\mathfrak{T}}_1$. So $\alpha'\circ\alpha\in\mathrm{KV}(\mathbf{k})$. One proves similarly that $\alpha^{-1}\in\mathrm{KV}(\mathbf{k})$. We have also proved that $\sigma_{\alpha'\circ\alpha} = \sigma_\alpha+\sigma_{\alpha'}$, i.e., $\mathrm{Duf}:\mathrm{KV}(\mathbf{k})\to u^2\mathbf{k}[[u]]$ is a group morphism.

6.2. **The torsor structure of** $\mathrm{SolKV}(\mathbf{k})$ **(proof of Proposition 2.3).** Let us prove that $\mathrm{KV}(\mathbf{k})$ acts on $\mathrm{SolKV}(\mathbf{k})$. For $\mu \in \mathrm{SolKV}(\mathbf{k})$, let $r_\mu := \mathrm{Duf}(\mu)$, so $r_\mu \in u^2\mathbf{k}[[u]]$, and $J(\mu) = \delta(r_\mu)$. For $\mu \in \mathrm{SolKV}(\mathbf{k})$, $\alpha \in \mathrm{KV}(\mathbf{k})$, we have $\mu\circ\alpha(X) \sim \mu(X) \sim e^x$, $\mu\circ\alpha(Y) \sim \mu(Y) \sim e^y$, $\mu \circ \alpha(XY) = \mu(XY) = e^{x+y}$. Moreover, $J(\mu \circ \alpha) = J(\mu) + \mu \cdot J(\alpha) = \delta(r_\mu) + \mu \cdot \tilde\delta(\sigma_\alpha) = \delta(r_\mu + \sigma_\alpha)$, where the last equality uses the identity $\delta(t) = \mu \cdot \tilde\delta(t)$ for $t \in \hat{\mathfrak{T}}_2$, which follows from $\mu(XY) = e^{x+y}$, $\mu(X) \sim e^x$, $\mu(Y) \sim e^y$. So $\mu \circ \alpha \in \mathrm{SolKV}(\mathbf{k})$. We have also proved that $r_{\mu\circ\alpha} = r_\mu + \sigma_\alpha$, so $\mathrm{Duf} : \mathrm{SolKV}(\mathbf{k}) \to u^2\mathbf{k}[[u]]$ is a morphism of torsors.

Let us now prove that the action of $\mathrm{KV}(\mathbf{k})$ on $\mathrm{SolKV}(\mathbf{k})$ is free and transitive. For $\mu, \mu' \in \mathrm{SolKV}(\mathbf{k})$, set $\alpha := \mu^{-1} \circ \mu'$; then $\alpha(X) \sim X$, $\alpha(Y) \sim Y$, $\alpha(XY) = XY$, and $J(\alpha) = J(\mu^{-1}) + \mu^{-1} \cdot J(\mu') = \mu^{-1} \cdot (J(\mu') - J(\mu))$ as $J(\mu^{-1}) = -\mu^{-1} \cdot J(\mu)$. Then $J(\alpha) = \mu^{-1} \cdot (\delta(r_{\mu'} - r_\mu)) = \tilde\delta(r_{\mu'} - r_\mu)$, where the last equality uses $\mu^{-1} \cdot \delta(t) = \tilde\delta(t)$ for $t \in \hat{\mathfrak{T}}_1$. So $\alpha \in \mathrm{KV}(\mathbf{k})$.

6.3. **Compatibilities of morphisms with group structures and actions (proof of Theorem 2.5).** We now show that: (a) $f \mapsto \alpha_f^{-1}$ is a group morphism $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$, (b) $g \mapsto a_g^{-1}$ is a group morphism $\mathrm{GRT}_1(\mathbf{k}) \to \mathrm{KRV}(\mathbf{k})$, (c) the map $\Phi \mapsto \mu_\Phi$ is compatible with the actions of these groups.

For this, we will show that

$$\mu_{f*\Phi} = \mu_\Phi \circ \alpha_f, \quad \mu_{\Phi*g} = a_g \circ \mu_\Phi. \tag{14}$$

We will check these identities on the first generator ($X$ or $x$), the proofs in the second case being similar.

The proofs go as follows:

$$\begin{aligned}
&\mu_{f*\Phi}(X) = (f * \Phi)(x, -x-y) \cdot e^x \cdot (same)^{-1}\\
&= f(\Phi(x, -x-y)e^x\Phi(x, -x-y)^{-1}, e^{-x-y})\Phi(x, -x-y) \cdot e^x \cdot (same)^{-1}\\
&= f(\mu_\Phi(X), \mu_\Phi(Y^{-1}X^{-1})) \cdot \mu_\Phi(X) \cdot (same)^{-1}\\
&= \mu_\Phi(f(X, Y^{-1}X^{-1}) \cdot X \cdot (same)^{-1}) = \mu_\Phi \circ \alpha_f(X)
\end{aligned}$$

and

$$\begin{aligned}
&\mu_{\Phi*g}(X) = (\Phi * g)(x, -x-y) \cdot e^x \cdot (same)^{-1}\\
&= \Phi(g(x, -x-y)xg(x, -x-y)^{-1}, -x-y)g(x, -x-y) \cdot e^x \cdot (same)^{-1}\\
&= \Phi(a_g(x), a_g(-x-y)) \cdot a_g(x) \cdot (same)^{-1}\\
&= a_g(\Phi(x, -x-y)x\Phi(x, -x-y)^{-1}) = a_g \circ \mu_\Phi(X).
\end{aligned}$$

The first part of (14) implies the following: (a) if $f \in \mathrm{GT}_1(\mathbf{k})$, then $\alpha_f \in \mathrm{KV}(\mathbf{k})$; (b) $\alpha_{f_1*f_2} = \alpha_{f_2} \circ \alpha_{f_1}$; (c) $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$ is compatible with the group morphism $f \mapsto \alpha_f^{-1}$.

Indeed, using the nonemptinesss of $M_1(\mathbf{k})$ (see [Dr]) we get $\alpha_f = \mu_\Phi^{-1} \circ \mu_{f*\Phi}$, which implies $\alpha_f \in \mathrm{KV}(\mathbf{k})$ according to Subsection 6.2, i.e., (a). Again using the nonemptiness of $M_1(\mathbf{k})$, we get $\alpha_{f_1*f_2} = \mu_\Phi^{-1} \circ \mu_{(f_1*f_2)*\Phi} = (\mu_\Phi^{-1} \circ \mu_{f_2*\Phi}) \circ (\mu_{f_2*\Phi}^{-1} \circ \mu_{f_1*(f_2*\Phi)}) = \alpha_{f_2} \circ \alpha_{f_1}$ (where we used $(f_1 * f_2) * \Phi = f_1 * (f_2 * \Phi)$), which proves (b). (c) is then tautological.

Similarly, the second part of (14) implies: (a) if $g \in \mathrm{GRT}_1(\mathbf{k})$, then $a_g \in \mathrm{KRV}(\mathbf{k})$; (b) $a_{g_1*g_2} = a_{g_2} \circ a_{g_1}$; (c) $M_1(\mathbf{k}) \to \mathrm{SolKV}(\mathbf{k})$ is compatible with the group morphism $g \mapsto a_g^{-1}$. All this proves Theorem 2.5.

It is easy to prove the identities $\alpha_{f_1*f_2}=\alpha_{f_2}\circ\alpha_{f_1}$, $a_{g_1*g_2}=a_{g_2}\circ a_{g_1}$ directly (i.e., not using the nonemptiness of $M_1(\mathbf{k})$): the verifications on the first generators ($X$ and $x$) are

$$\begin{aligned}&\alpha_{f_1*f_2}(X)=(f_1*f_2)(X,Y^{-1}X^{-1})\cdot X\cdot(same)^{-1}\\&=f_1(f_2(X,Y^{-1}X^{-1})Xf_2(X,Y^{-1}X^{-1})^{-1},Y^{-1}X^{-1})f_2(X,Y^{-1}X^{-1})\cdot X\cdot(same)^{-1}\\&=f_1(\alpha_{f_2}(X),\alpha_{f_2}(Y^{-1}X^{-1}))\cdot\alpha_{f_2}(X)\cdot(same)^{-1}\\&=\alpha_{f_2}(f_1(X,Y^{-1}X^{-1})\cdot X\cdot(same)^{-1})=\alpha_{f_2}\circ\alpha_{f_1}(X),\end{aligned}$$

and

$$\begin{aligned}&a_{g_1*g_2}(x)=(g_1*g_2)(x,-x-y)\cdot x\cdot(same)^{-1}\\&=g_1(g_2(x,-x-y)xg_2(x,-x-y)^{-1},-x-y)g_2(x,-x-y)\cdot x\cdot(same)^{-1}\\&=g_1(a_{g_2}(x),a_{g_2}(-x-y))\cdot a_{g_2}(x)\cdot(same)^{-1}\\&=a_{g_2}(g_1(x,-x-y)xg_1(x,-x-y)^{-1})=a_{g_2}\circ a_{g_1}(x).\end{aligned}$$

**Remark 6.1.** The Lie algebra morphism corresponding to $g\mapsto a_g^{-1}$ is the morphism $\nu:\mathfrak{grt}_1\to\mathfrak{krv}$ from [AT], given by $\psi(x,y)\mapsto[\![\psi(x,-x-y),\psi(y,-x-y)]\!]$.

6.4. **Torsor properties of the Duflo formal series (proof of Proposition 2.6).** We have already proved that $M_1(\mathbf{k})\to\mathrm{SolKV}(\mathbf{k})$, and $\mathrm{SolKV}(\mathbf{k})\stackrel{\mathrm{Duf}}{\to}u^2\mathbf{k}[[u]]$ is a morphism of torsors. On the other hand, it follows from [E] that $M_1(\mathbf{k})\stackrel{\Phi\mapsto\log\Gamma_\Phi}{\to}\{r\in u^2\mathbf{k}[[u]]|r_{ev}(u)=-\frac{u^2}{24}+...\}$ is a morphism of torsors and from Proposition 2.2 that the diagram of Proposition 2.6) commutes.

For later use, let us make the group morphism $\mathrm{GT}_1(\mathbf{k})\to u^3\mathbf{k}[[u^2]]$ underlying $\Phi\mapsto\log\Gamma_\Phi$ explicit.

**Lemma 6.2.** *For $f\in\mathrm{GT}_1(\mathbf{k})$, there is a unique $\Gamma_f\in\exp(u^3\mathbf{k}[[u^2]])$ such that*

$$[\log f(e^a,e^b)]=1-\frac{\Gamma_f(-\overline{a})\Gamma_f(-\overline{b})}{\Gamma_f(-\overline{a}-\overline{b})};$$

*here we use the isomorphism $\hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2''\simeq\overline{a}\overline{b}\mathbf{k}[[\overline{a},\overline{b}]]$ given by (class of $(\mathrm{ad}\,a)^k(\mathrm{ad}\,b)^l([a,b])$) $\leftrightarrow\overline{a}^{k+1}\overline{b}^{l+1}$. The map $\mathrm{GT}_1(\mathbf{k})\to u^3\mathbf{k}[[u^2]]$, $f\mapsto\log\Gamma_f$ is a group morphism and $\Gamma_{f*\Phi}=\Gamma_f\Gamma_\Phi$ for any $f\in\mathrm{GT}_1(\mathbf{k})$, $\Phi\in M_1(\mathbf{k})$.*

*Proof.* The map $\mathfrak{f}_2\to\mathbf{k}[\overline{a},\overline{b}]$, $\psi\mapsto(b\partial_b\psi)^{ab}$ also induces an isomorphism $\hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2''\simeq\overline{a}\overline{b}\mathbf{k}[[\overline{a},\overline{b}]]$, which takes the class $(\mathrm{ad}\,a)^k(\mathrm{ad}\,b)^l([a,b])$ to $(-1)^{k+l+1}\overline{a}^{k+1}\overline{b}^{l+1}$. So for $\psi\in\hat{\mathfrak{f}}_2'$, we have $(b\partial_b\psi)^{ab}(\bar a,\bar b)=-[\psi](-\bar a,-\bar b)$ (where $\psi\mapsto[\psi]$ is the map $\hat{\mathfrak{f}}_2'\to\hat{\mathfrak{f}}_2'/\hat{\mathfrak{f}}_2''\simeq\overline{a}\overline{b}\mathbf{k}[[\overline{a},\overline{b}]]$).

So (6) may be rewritten

$$[\log\Phi](\overline{a},\overline{b})=1-\frac{\Gamma_\Phi(-\overline{a}-\overline{b})}{\Gamma_\Phi(-\overline{a})\Gamma_\Phi(-\overline{b})}.$$

If now $\psi,\alpha\in\hat{\mathfrak{f}}_2'$, we have $\psi(e^{-\alpha}ae^\alpha,b)\in\hat{\mathfrak{f}}_2'$ and $[\psi(e^{-\alpha}ae^\alpha,b)]=(1-[\alpha(a,b)])[\psi(a,b)]$. Indeed, when $\psi(a,b)=(\mathrm{ad}\,a)^k(\mathrm{ad}\,b)^l([a,b])$, one checks that the part of $\psi(e^{-\alpha}ae^\alpha,b)$ containing $\alpha$ more than twice lies in $\hat{\mathfrak{f}}_2''$, and the part containing it once has the same class as $(\mathrm{ad}\,a)^k(\mathrm{ad}\,b)^l([[-\alpha,a],b])$.

If now $f\in\mathrm{GT}_1(\mathbf{k})$, we have $(f*\Phi)(a,b)=\Phi(a,b)f(\Phi^{-1}(a,b)e^a\Phi(a,b),e^b)$, so

$$\begin{aligned}[\log(f*\Phi)(a,b)]&=[\log\Phi(a,b)]+[\log f(\Phi^{-1}(a,b)e^a\Phi(a,b),e^b)]\\&=[\log\Phi(a,b)]+[\log f(e^a,e^b)]-[\log\Phi(a,b)][\log f(e^a,e^b)].\end{aligned}$$

so

$$1-[\log(f*\Phi)(a,b)]=(1-[\log\Phi(a,b)])(1-[\log f(e^a,e^b)]).\qquad(15)$$

If fix $\Phi_0 \in M_1(\mathbf{k})$ and set $\Gamma_f(u) := \Gamma_{f*\Phi_0}(u)/\Gamma_{\Phi_0}(u)$, then we get

$$1 - [\log f(e^a, e^b)] = \frac{\Gamma_f(-\overline{a})\Gamma_f(-\overline{b})}{\Gamma_f(-\overline{a} - \overline{b})}$$

as wanted. Moreover, (15) implies that $\Gamma_{f*\Phi} = \Gamma_f \Gamma_\Phi$, which also implies that $f \mapsto \Gamma_f$ is a group morphism. □

## 7. Direct construction of the map $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{KV}(\mathbf{k})$

We will now sketch a proof of $(f \in \mathrm{GT}_1(\mathbf{k})) \Rightarrow (\alpha_f \in \mathrm{KV}(\mathbf{k}))$, independent of the nonemptiness of $M_1(\mathbf{k})$.

7.1. **Action of $\mathrm{GT}_1(\mathbf{k})$ on completed braid groups.** Let $\mathcal{C}$ be a b.m.c. We denote by $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ and $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ the braiding and associativity constraints. For $O \in \mathrm{Ob}(\mathbf{PaB})$ of length $n$ and any $X_1, ..., X_n \in \mathrm{Ob}(\mathcal{C})$, we construct the tensor product $O(X_1, ..., X_n)$ of $X_1, ..., X_n$ with parenthesization $O$. We say that $\mathcal{C}$ is prounipotent if for any $X_1, ..., X_n$ and any $O$, the image of $\mathrm{PB}_n \to \mathrm{Aut}_{\mathcal{C}}(O(X_1, ..., X_n))$ is prounipotent (it suffices to require this for a given $O$). If $\mathcal{C}$ is a prounipotent b.m.c. and $f \in \mathrm{GT}_1(\mathbf{k})$, we construct a new b.m.c. ${}^f\mathcal{C}$ as follows: ${}^f\mathcal{C}$ is the same as $\mathcal{C}$ at the level of objects and morphisms, the composition and the tensor product of morphisms are not modified, but the braiding and associativity constraints are modified as follows:

$$\beta'_{X,Y} = \beta_{X,Y}, \quad a'_{X,Y,Z} = a_{X,Y,Z} \circ f(\beta_{YX}\beta_{XY}, a^{-1}_{X,Y,Z} \circ \beta_{ZY}\beta_{YZ} \circ a_{X,Y,Z}).$$

We then have ${}^{f_1}({}^{f_2}\mathcal{C}) = {}^{f_1 * f_2}\mathcal{C}$. Moreover, the action of $\mathrm{GT}_1(\mathbf{k})$ on BMC is functorial, so a tensor functor $\phi : \mathcal{C} \to \mathcal{D}$ and $f \in \mathrm{GT}_1(\mathbf{k})$ give rise to ${}^f\phi : {}^f\mathcal{C} \to {}^f\mathcal{D}$. Note that for $O, O' \in \mathrm{Ob}(\mathcal{C})$, and under the identifications ${}^f\mathcal{C}(O, O') = \mathcal{C}(O, O')$, ${}^f\mathcal{D}(\phi(O), \phi(O')) = \mathcal{D}(\phi(O), \phi(O'))$, the map ${}^f\phi(O, O') : {}^f\mathcal{C}(O, O') \to {}^f\mathcal{D}(\phi(O), \phi(O'))$ coincides with $\phi(O, O') : \mathcal{C}(O, O') \to \mathcal{D}(\phi(O), \phi(O'))$.

Let $\mathbf{PaB_k}$ be the completion of $\mathbf{PaB}$ obtained by replacing each group $\mathrm{B}_n$ by its completion $\mathrm{B}_n(S_n, \mathbf{k})$ relative to the morphism $\mathrm{B}_n \to S_n$. By universal properties, we have a unique morphism $\phi_f : \mathbf{PaB} \to {}^f\mathbf{PaB}$ which is the identity on objects. If then $f_1, f_2 \in \mathrm{GT}_1(\mathbf{k})$, we have

$$(16) \qquad {}^{f_1}\phi_{f_2} \circ \phi_{f_1} = \phi_{f_1 * f_2};$$

indeed, both terms are tensor functors $\mathbf{PaB_k} \to {}^{f_1 * f_2}\mathbf{PaB_k}$ which are the identity on objects.

If now $O \in \mathrm{Ob}(\mathbf{PaB})$ has length $n$, $\phi_f$ gives rise to a group morphism $\phi_f(O) : \mathbf{PaB_k}(O) \to {}^f\mathbf{PaB_k}(O)$. We denote by

$$\tilde\alpha_f^O : \mathrm{B}_n(S_n, \mathbf{k}) \to \mathrm{B}_n(S_n, \mathbf{k})$$

the group endomorphism derived from $\phi_f(O)$ and the identifications $\mathbf{PaB_k}(O) = {}^f\mathbf{PaB_k}(O) = \mathrm{B}_n(S_n, \mathbf{k})$. Identity (16) and the identification of ${}^{f_1}\tilde\alpha^O_{f_2}$ with $\tilde\alpha^O_{f_2}$ imply

$$\tilde\alpha^O_{f_2} \circ \tilde\alpha^O_{f_1} = \tilde\alpha^O_{f_1 * f_2},$$

so we have a group antimorphism $\mathrm{GT}_1(\mathbf{k}) \to \mathrm{Aut}(\mathrm{B}_n(S_n, \mathbf{k}))$, $f \mapsto \tilde\alpha^O_f$.

It is easy to see that we have a commutative diagram $\begin{array}{ccc} \mathrm{B}_n(S_n, \mathbf{k}) & \stackrel{\tilde\alpha_f^O}{\to} & \mathrm{B}_n(S_n, \mathbf{k}) \\ \searrow & & \swarrow \\ & S_n & \end{array}$ so $\tilde\alpha^O_f$ restricts to an automorphism $\tilde\alpha^O_f \in \mathrm{Aut}(\mathrm{PB}_n(\mathbf{k}))$.

If now $O, O' \in \mathrm{Ob}(\mathbf{PaB})$ have length $n$, then $\mathrm{can}_{O,O'} \in \mathbf{PaB_k}(O, O')$ is the morphism corresponding to the trivial braid. Then $\phi_f(\mathrm{can}_{O,O'}) \circ \mathrm{can}^{-1}_{O,O'} \in \mathbf{PaB_k}(O)$. Let $f^{O,O'} \in \mathrm{PB}_n(\mathbf{k})$

be the image of this element. Since the diagram $\begin{array}{ccc} \mathbf{PaB}(O) & \stackrel{x\mapsto \mathrm{can}_{O,O'}\circ x\circ \mathrm{can}_{O,O'}^{-1}}{\to} & \mathbf{PaB}(O') \\ \searrow & & \swarrow \\ & \mathrm{B}_n(S_n,\mathbf{k}) & \end{array}$

commutes, we have

$$\tilde\alpha_f^{O'} = \mathrm{Inn}(f^{O,O'})\circ\tilde\alpha_f^O. \tag{17}$$

7.2. **Actions of** $\mathrm{GT}_1(\mathbf{k})$ **on free groups.** Let us index the generators of $\mathrm{PB}_n(\mathbf{k})$ by $x_{ij}$, $0\leq i<j\leq n-1$. Recall that the subgroup of $\mathrm{PB}_n(\mathbf{k})$ generated by $x_{01},...,x_{0,n-1}$ is isomorphic to $\mathrm{F}_{n-1}(\mathbf{k})$. We set $X_i = x_{0i}$ for $i=1,...,n-1$.

**Proposition 7.1.** *Each* $\tilde\alpha_f^O$ *restricts to an automorphism* $\alpha_f^O\in\mathrm{Aut}(\mathrm{F}_{n-1}(\mathbf{k}))$*, such that for any* $i$*,* $\alpha_f^O(X_i)\sim X_i$*.*

*Proof.* Let us index the letters of $O$ by $0,...,n-1$. For $i=1,...,n-1$, let $O_i$ be an object of $\mathbf{PaB}$ of length $n$, in which the letters $i-1$ and $i$ appear as $...(\bullet\bullet)....$. We have $X_i = (\sigma_0...\sigma_{i-2})^{-1}\sigma_{i-1}^2\sigma_0...\sigma_{i-2}$. We have $\tilde\alpha_f^O(\sigma_0...\sigma_{i-2}) = \sigma_0...\sigma_{i-2}\cdot p_i$, where $p_i\in\mathrm{PB}_n(\mathbf{k})$. On the other hand, $\tilde\alpha_f^O(\sigma_{i-1}) = f^{O,O_i}\tilde\alpha_f^O(\sigma_{i-1})(f^{O,O_i})^{-1}$ and $\tilde\alpha_f^O(\sigma_{i-1}) = \sigma_{i-1}$ as $\mathrm{B}_n\simeq\mathbf{PaB}(O_i)$ takes $\sigma_{i-1}$ to $\mathrm{id}_\bullet^{\otimes i-1}\otimes\beta_{\bullet,\bullet}\otimes\mathrm{id}_\bullet^{\otimes n-i-2}$. So

$$\tilde\alpha_f^O(\sigma_{i-1}^2) = f^{O,O_i}\sigma_{i-1}^2(f^{O,O_i})^{-1},$$

with $\alpha_f^{O,O_i}\in\mathrm{PB}_n(\mathbf{k})$. Then

$$\begin{aligned}\tilde\alpha_f^O(X_i) &= (\sigma_0...\sigma_{i-2}p_i)^{-1}f^{O,O_i}\sigma_{i-1}^2(f^{O,O_i})^{-1}\sigma_0...\sigma_{i-2}p_i \\ &= p_i^{-1}(\sigma_0...\sigma_{i-2})^{-1}f^{O,O_i}(\sigma_0...\sigma_{i-2})\cdot X_i\cdot(\mathit{same})^{-1}.\end{aligned} \tag{18}$$

As $p_i^{-1}(\sigma_0...\sigma_{i-2})^{-1}f^{O,O_i}(\sigma_0...\sigma_{i-2})$ belongs to $\mathrm{PB}_n(\mathbf{k})$, and as $\mathrm{PB}_n(\mathbf{k})$ acts on $\mathrm{F}_{n-1}(\mathbf{k})$ by tangential automorphisms, we obtain that $\tilde\alpha_f^O(X_i)$ lies in $\mathrm{F}_{n-1}(\mathbf{k})$ and is conjugated in $\mathrm{F}_{n-1}(\mathbf{k})$ to $X_i$. ☐

Similarly to Proposition 3.2, one can prove:

**Proposition 7.2.** *If* $O=\bullet\otimes\bar O$*, where* $\bar O\in\mathrm{Ob}(\mathbf{PaB})$*, then* $\alpha_f^O(X_1...X_{n-1}) = X_1...X_{n-1}$*.*

We then have

$$\alpha_f^{O'} = \mathrm{Ad}(f^{O,O'})\circ\alpha_f^O;$$

this is an identity in $\mathrm{Aut}(\mathrm{F}_{n-1}(\mathbf{k}))$, where $\mathrm{Ad}(\alpha_f^{O,O'})$ is not necessarily inner.

We also record the identities

$$\tilde\mu_{f*\Phi}^O = \tilde\mu_\Phi^O\circ\alpha_f^O,\quad \mu_{f*\Phi}^O = \mu_\Phi^O\circ\alpha_f^O. \tag{19}$$

7.3. **The map** $\mathrm{GT}_1(\mathbf{k})\to\mathrm{KV}(\mathbf{k})$**.** Let us fix an element $f\in\mathrm{GT}_1(\mathbf{k})$ and denote $\tilde\alpha_f^O$, $\alpha_f^O$ simply by $\tilde\alpha_O$, $\alpha_O$.

As in Subsection 4.5, one proves that

$$\begin{array}{ccc} \mathrm{PB}_n(\mathbf{k}) & \stackrel{1,2,...,\widetilde{ii+1},...,n}{\to} & \mathrm{PB}_{n+1}(\mathbf{k}) \\ {\scriptstyle\alpha_O}\downarrow & & \downarrow{\scriptstyle\alpha_{O^{(i)}}} \\ \mathrm{PB}_n(\mathbf{k}) & \stackrel{1,2,...,\widetilde{ii+1},...,n}{\to} & \mathrm{PB}_{n+1}(\mathbf{k}) \end{array} \tag{20}$$

commutes. Using Proposition A.3, one then proves

$$\alpha_{O^{(i)}} = \alpha_O^{1,...,\widetilde{ii+1},...,n}\circ\alpha_{\bullet(\bullet\bullet)}^{i,i+1}. \tag{21}$$

Similarly to Proposition 5.1, one proves that

1) $\alpha_{\bullet(\bullet\bullet)} = \alpha_f$.
2) $f^{\bullet((\bullet\bullet)\bullet),\bullet(\bullet(\bullet\bullet))} = f(x_{12},x_{23})$.

As in Subsection 5.1, one proves that this implies

$$\mathrm{Ad}\, f(x_{12},x_{23})\circ\alpha_f^{\widetilde{12},3}\circ\alpha_f^{1,2}=\alpha_f^{1,\widetilde{23}}\circ\alpha_f^{2,3}. \tag{22}$$

As in Subsection 5.2, one can give three proofs of the fact that $\alpha_f(XY)=XY$. Similarly to Subsection 5.3, one then proves that identity (22) then implies that $J(\alpha_f)$ is a $\tilde\delta$-coboundary.

Let us explain this proof in some detail. Since $J(\mathrm{Ad}\, f(x_{12},x_{23}))=0$ and $J(\alpha_f^{\widetilde{12},3})=J(\alpha_f)^{\widetilde{12},3}$, we get by applying $J$ to (22)

$$\mathrm{Ad}\, f(x_{12},x_{23})\cdot J(\alpha_f)^{\widetilde{12},3}+\big(\mathrm{Ad}\, f(x_{12},x_{23})\circ\alpha_f^{\widetilde{12},3}\big)\cdot J(\alpha_f)^{1,2}=J(\alpha_f)^{\widetilde{12},3}+J(\alpha_f)^{2,3}.$$

Applying the inverse of (22), we get

$$(\alpha_f^{1,2})^{-1}\cdot(\alpha_f^{-1}\cdot J(\alpha_f))^{\widetilde{12},3}+(\alpha_f^{-1}\cdot J(\alpha_f))^{1,2}=(\alpha_f^{2,3})^{-1}\cdot(\alpha_f^{-1}\cdot J(\alpha_f))^{1,\widetilde{23}}+(\alpha_f^{-1}\cdot J(\alpha_f))^{2,3}$$

Now $\alpha_f(XY)=XY$ implies that $\alpha_f^{1,2}\cdot t^{\widetilde{12},3}=t^{\widetilde{12},3}$ and similarly with $1,\tilde{23}$, so $\tilde\delta(\alpha_f^{-1}\cdot J(\alpha_f))=0$. As $\hat{\mathfrak{T}}_1\stackrel{\tilde\delta}{\to}\hat{\mathfrak{T}}_2\to\ldots$ is acyclic in degree 2, there exists $\beta\in\hat{\mathfrak{T}}_1$ with valuation $\geq 2$ such that $\alpha_f^{-1}\cdot J(\alpha_f)=\tilde\delta(\beta)$, so $J(\alpha_f)=\alpha_f\cdot\tilde\delta(\beta)$. Now $\alpha_f(XY)=XY$, $\alpha_f(X)\sim X$, $\alpha_f(Y)\sim Y$ imply that $\alpha_f\cdot\tilde\delta(\beta)=\tilde\delta(\beta)$, so $J(\alpha_f)=\tilde\delta(\beta)$. It follows that $J(\alpha_f)$ has the form $\tilde\delta(\beta)=\langle\beta(\log(e^xe^y))-\beta(x)-\beta(y)\rangle$.

**Remark 7.3.** (22) can also be proved directly, checking the identity on each of the generators of $\mathrm{F}_3(\mathbf{k})$ and using only the duality, hexagon and pentagon relations. This proof then extends to the profinite and pro-$l$ cases.

## 8. The Jacobians of $\mu_{\Phi,O}$ and $\alpha_f^O$

8.1. **Telescopic formulas.** If $O\in\mathrm{Ob}(\mathbf{PaB})$ has the form $O=\bullet\otimes O'$, with $|O'|=n$, then one proves by using (4) that $\mu_O$ expresses directly in terms of $\mu_\Phi$, for example

$$\mu_{\bullet((((\bullet\bullet)(\bullet\bullet))(\bullet(\bullet\bullet)))(\bullet\bullet))}=\mu_\Phi^{1234567,89}\mu_\Phi^{1234,567}\mu_\Phi^{8,9}\mu_\Phi^{12,34}\mu_\Phi^{5,67}\mu_\Phi^{1,2}\mu_\Phi^{3,4}\mu_\Phi^{6,7}.$$

The general formula is

$$\mu_{\bullet\otimes O'}=\prod_{n\geq 0}\prod_{\nu\in N(T'),d(\nu)=n}\mu_\Phi^{L(\nu),R(\nu)};$$

here $T'$ is the binary planar rooted tree underlying $O'$; $N(T')$ is the set of its nodes; $d(\nu)$ is the degree of $\nu$ (distance to the root of the tree); $L(\nu)$, $R(\nu)$ is the set of left and right leaves of $\nu$ (these are disjoints subsets of $\{1,\ldots,n\}$). The first product is taken according to increasing values of $n$ (the order in the second product does not matter as the arguments of this product commute with each other). Here is the tree corresponding to the above example (Figure 3):

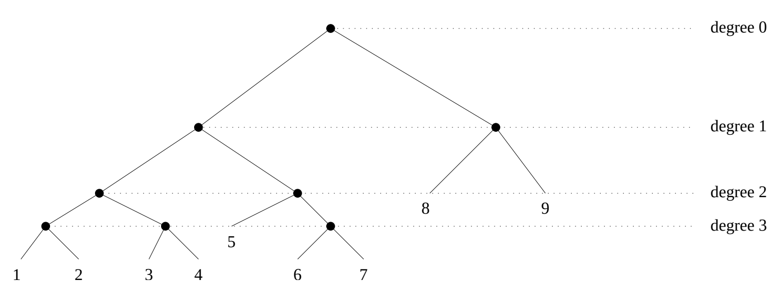


FIGURE 3. There are 8 nodes

Similarly, using (21), one proves that for $f \in \mathrm{GT}_1(\mathbf{k})$, we have

$$\alpha_f^{\bullet\otimes O'} = \prod_{n\geq 0} \prod_{\nu\in N(T'),d(\nu)=n} \alpha_f^{\widetilde{L(\nu)},\widetilde{R(\nu)}}.$$

8.2. **Computation of Jacobians.** Let $\mu_n := \mu_{\bullet(\bullet\ldots(\bullet\bullet))}$. Then:

**Proposition 8.1.** $J(\mu_n) = \langle \sum_{i=1}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=1}^n x_i)\rangle$.

(We identified $\mu_n$ with its composition with $e^{x_i} \mapsto X_i$, which belongs to $\mathrm{TAut}_n$.)

*Proof.* We have $\mu_n = \mu_\Phi^{1,2\ldots n} \circ \mu_\Phi^{2,3\ldots n} \circ \ldots \circ \mu_\Phi^{n-1,n}$. One then proves by descending induction on $k$ that $J(\mu_\Phi^{k,k+1\ldots n} \circ \ldots \circ \mu_\Phi^{n-1,n}) = \langle \sum_{i=k}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=k}^n x_i)\rangle$, using the fact that the action of $\mu_\Phi^{k,k+1\ldots n}$ on the various $\langle \log\Gamma_\Phi(x_i)\rangle$ as well as on $\langle\log\Gamma_\Phi(\sum_{i=k}^n x_i)\rangle$ is trivial. ∎

If now $O \in \mathrm{Ob}(\mathbf{PaB})$ is arbitrary with with $|O| = n+1$, then:

**Proposition 8.2.** $J(\mu_{\Phi,O}) = J(\mu_n) = \langle \sum_{i=1}^n \log\Gamma_\Phi(x_i) - \log\Gamma_\Phi(\sum_{i=1}^n x_i)\rangle$.

*Proof.* We have $\mu_O = \mathrm{Ad}\,\Phi_{O_n,O} \circ \mu_n$, where $O_n = \bullet(\ldots(\bullet\bullet))$. We then use the cocycle property of $J$, the above formula for $J(\mu_n)$, the fact that $J(\mathrm{Ad}\,g) = 0$ for $g \in \exp(\hat{\mathfrak{t}}_{n+1})$, and the following lemma:

**Lemma 8.3.** *If* $g \in \exp(\hat{\mathfrak{t}}_{n+1})$*, then* $(\mathrm{Ad}\,g)(x_1 + \ldots + x_n) \sim x_1 + \ldots + x_n$.

*Proof of Lemma.* Decompose $a \in \mathfrak{t}_{n+1}$ as $a_0 + a_1^{1,2,\ldots,n}$, with $a_0 \in \mathfrak{f}_n$ and $a_1 \in \mathfrak{t}_n$ (the map $a_1 \mapsto a_1^{1,2,\ldots,n}$ is the injection $\mathfrak{t}_n \to \mathfrak{t}_{n+1}$, $t_{ij} \mapsto t_{ij}$). Then $[t_{ij}, x_1 + \ldots + x_n] = 0$ for $i,j \in \{1,\ldots,n\}$, so $[a_1^{1,2,\ldots,n}, x_1 + \ldots + x_n] = 0$, so $[a, x_1 + \ldots + x_n] = [a_0, x_1 + \ldots + x_n]$. It follows that if $g \in \exp(\hat{\mathfrak{t}}_{n+1})$, there exists $x_g \in \exp(\hat{\mathfrak{f}}_n)$ such that $(\mathrm{Ad}\,g)(x_1 + \ldots + x_n) = g(x_1 + \ldots + x_n)g^{-1}$. ∎ ∎

We then have:

**Proposition 8.4.** $J(\alpha_f^O) = \langle \sum_{i=1}^n \log\Gamma_f(\log X_i) - \log\Gamma_f(\log \prod_{i=1}^n X_i)\rangle$.

*Proof.* Fix $\Phi \in M_1(\mathbf{k})$. We have $\mu_{f*\Phi}^O = \mu_\Phi^O \circ \alpha_f^O$, so $J(\mu_{f*\Phi}^O) = J(\mu_\Phi^O) + \mu_\Phi^O \circ J(\alpha)_f^O$. It follows that $\mu_\Phi^O \circ J(\alpha)_f^O = \langle \sum_{i=1}^n \log\Gamma_f(x_i) - \log\Gamma_f(\sum_{i=1}^n x_i)\rangle$. The result then follows from $\mu_{Phi}^O(X_i) \sim e^{x_i}$, $\mu_\Phi^O(X_1 \ldots X_n) \sim e^{x_1+\ldots+x_n}$. ∎

**Remark 8.5.** In [AT], the Lie subalgebra $\mathfrak{sder}_n \subset \mathfrak{tder}_n$ of special derivations (normalized special in the terms of Ihara) was introduced: $\mathfrak{sder}_n = \{u \in \mathfrak{tder}_n | u(x_1 + \ldots + x_n) = 0\}$. Let $\tilde{\mathfrak{sder}}_n$ be the intermediate Lie algebra $\tilde{\mathfrak{sder}}_n = \{u \in \mathfrak{tder}_n | \exists u_0 \in \mathfrak{f}_{n-1} | u(x_1 + \ldots + x_n) = [u_0, x_1 + \ldots + x_n]\}$ (special derivations in Ihara's terms). So $\mathfrak{sder}_n \subset \tilde{\mathfrak{sder}}_n \subset \mathfrak{tder}_n$. Then Lemma 8.3 says that we have a diagram

$$\begin{array}{ccccc} \mathfrak{t}_n & \to & \mathfrak{sder}_n & & \\ \downarrow & & \downarrow & & \\ \mathfrak{t}_{n+1} & \to & \tilde{\mathfrak{sder}}_n & \hookrightarrow & \mathfrak{tder}_n \end{array}$$

**Remark 8.6.** Set $\mathrm{SolKV}_n(\mathbf{k}) := \{\mu_n \in \mathrm{TAut}_n | \mu_n(e^{x_1}\ldots e^{x_n}) = e^{x_1+\cdots+x_n}$ and $\exists r \in u^2\mathbf{k}[[u]] | J(\mu_n) = \langle r(\sum_i x_i) - \sum_i r(x_i)\rangle\}$. This is a torsor under the action of the groups $\mathrm{KV}_n(\mathbf{k}) := \{\alpha_n \in \mathrm{TAut}_n | \alpha_n(e^{x_1}\ldots e^{x_n}) = e^{x_1}\ldots e^{x_n}$ and $\exists\sigma \in u^2\mathbf{k}[[u]] | J(\alpha) = \langle \sigma(\log e^{x_1}\ldots e^{x_n}) - \sum_i \sigma(x_i)\rangle\}$ and $\mathrm{KRV}_n(\mathbf{k})$ defined similarly (replacing $e^{x_1}\ldots e^{x_n}$ by $e^{x_1+\cdots+x_n}$). These are prounipotent groups; the Lie algebra of $\mathrm{KRV}_n(\mathbf{k})$ is $\mathfrak{krv}_n := \{u \in \mathfrak{tder}_n | a(\sum_i x_i) = 0$ and $\exists s \in u^2\mathbf{k}[[u]] | j(a) = \langle s(\sum_i x_i) - \sum_i s(x_i)\rangle\}$. It contains as a Lie subalgebra $\mathfrak{krv}_n^0 := \{a \in \mathfrak{krv}_n | s = 0\}$, which is denoted $\mathfrak{kv}_n$ in [AT]. One can prove that if $|O'| = n$ and $O = \bullet \otimes O'$, the map $M_1(\mathbf{k}) \to \mathrm{SolKV}_n(\mathbf{k})$, $\Phi \mapsto \mu_{\Phi,O}$ is a morphism of torsors.

## 9. Analytic aspects

In this section, the base field $\mathbf{k}$ is $\mathbb{R}$ or $\mathbb{C}$.

9.1. **Analytic germs.** We set $\mathbb{R}_+\{\{x\}\} := \{f \in \mathbb{R}_+[[x]] | f$ has posititive radius of convergence$\}$ and $\mathbb{R}_+\{\{x\}\}_0 := \{f \in \mathbb{R}_+\{\{x\}\} | f(0) = 0\}$. If $f, g \in \mathbb{R}_+[[r]]$, we write $f \preceq g$ iff $g - f \in \mathbb{R}_+[[r]]$. We define $f \preceq g$ similarly when $f, g \in \mathbb{R}_+[[r_1, ..., r_n]]$.

Let $V, E$ be finite dimensional vector spaces and let $|.|_V, |.|_E$ be norms on $V, E$. The space of $E$-valued formal series on $V$ is $E[[V]] = \{f = \sum_{n\geq 0} f_n, f_n \in S^n(V^*) \otimes E\}$. For $f_n \in S^n(V^*) \otimes E$, viewed as an homogeneous polynomial $V \to E$, we set $|f_n| := \sup_{v\neq 0}(|f_n(v)|_E/|v|_V^n)$. An analytic germ on $V$ (at the neighborhood of 0) is a series $f \in E[[V]]$, such that $|f|(r) := \sum_{n\geq 0} |f_n| r^n \in \mathbb{R}_+\{\{r\}\}$. We denote by $E\{\{V\}\} \subset E[[V]]$ the subspace of analytic germs, and by $E\{\{V\}\}_0 \subset E[[V]]_0$ the subspace defined by $f_0 = 0$.

If $f \in E\{\{V\}\}$ and $\alpha = \sum_{n\geq 0} \alpha_n r^n \in \mathbb{R}_+[[r]]_0$, we say that $\alpha$ is a dominating series for $f$ is $|f_n| \leq \alpha_n$ for any $n$; we write this as $|f(v)|_E \preceq \alpha(|v|_V)$.

If $V_1, ..., V_k$ are finite dimensional vector spaces with norms $|.|_{V_1}, ..., |.|_{V_k}$, then we equip $V_1 \oplus ... \oplus V_k$ with the norm $|(v_1, ..., v_k)| := \sup_k |v_i|_{V_i}$. If $f$ is an analytic germ $V_1 \oplus ... \oplus V_k \to E$, we decompose $f = \sum_{\mathbf{n}\in\mathbb{N}^k} f_{\mathbf{n}}$, where $f_{\mathbf{n}} : V_1 \times ... \times V_k \to E$ is the $\mathbf{n}$-multihomogeneous component of $f$. We then set

$$|f_{\mathbf{n}}| := \sup_{(x_1,...,x_k)\in\prod_i (V_i - \{0\})} |f_{\mathbf{n}}(x_1, ..., x_k)|_E / |x_1|_{V_1}^{n_1} ... |x_k|_{V_k}^{n_k}.$$

Then $f$ is an analytic germ iff $|f|(r_1, ..., r_n) := \sum_{\mathbf{n}} |f_{\mathbf{n}}| r_1^{n_1} ... r_k^{n_k} \in \mathbb{R}_+[[r_1, ..., r_k]]$ converges in a polydisc. If $\alpha = \sum_{n_1,...,n_k\geq 0} \alpha_{n_1,...,n_k} r_1^{n_1} ... r_k^{n_k} \in \mathbb{R}_+[[r_1, ..., r_k]]$, we write $|f(v_1, ..., v_k)|_E \preceq \alpha(|v_1|_{V_1}, ..., |v_k|_{V_k})$ if for each $\mathbf{n}$, $|f_{\mathbf{n}}(v_1, ..., v_k)|_E \leq \alpha_{\mathbf{n}}(|v_1|_{V_1}, ..., |v_k|_{V_k})$.

Let now $\mathfrak{g}$ be a finite dimensional Lie algebra; let $|.|$ be a norm on $\mathfrak{g}$; let $M > 0$ be such that the identity $|[x, y]| \leq M|x||y|$ holds.

The specialization to $\mathfrak{g}$ of the Campbell–Baker–Hausdorff series is a series $x * y = \text{cbh}(x, y) \in \mathfrak{g}[[\mathfrak{g} \times \mathfrak{g}]]_0$.

**Lemma 9.1.** *1) The CBH series is an analytic germ $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$; we have $|x*y| \preceq \frac{1}{M} f(M(|x|+|y|))$, where $f(u) = \int_0^u -\frac{\ln(2-e^v)}{v} dv$.*

*2) $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto e^{\text{ad}\, x}(y)$ is an analytic germ, and $|e^{\text{ad}\, x}(y)| \preceq e^{M|x|}|y|$.*

*Proof.* 1) is proved as in [Bk], not making use of the final majorization $\frac{1}{r+s} \leq 1$. Using Dynkin's formula, one can prove that 2) follows from $|(\text{ad}\, x)^n(y)| \leq M^n |x|^n |y|$. □

9.2. $\text{TAut}_n^{an}(\mathfrak{g})$ **and** $\mathfrak{tder}_n^{an}(\mathfrak{g})$**.** We set $\text{TAut}_n(\mathfrak{g}) := \{(a_1, ..., a_n) | a_i \in \mathfrak{g}[[\mathfrak{g}^n]]_0\}$ and define on this set a product by $(a_1, ..., a_n)(b_1, ..., b_n) := (c_1, ..., c_n)$, where

$$c_i(x_1, ..., x_n) := b_i(e^{\text{ad}\, a_1(x_1,...,x_n)}(x_1), ..., e^{\text{ad}\, a_n(x_1,...,x_n)}(x_n)) * a_i(x_1, ..., x_n).$$

This equips $\text{TAut}_n(\mathfrak{g})$ with a group structure. We set $\text{TAut}_n^{an}(\mathfrak{g}) := \{(a_1, ..., a_n) | a_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0\}$.

**Proposition 9.2.** $\text{TAut}_n^{an}(\mathfrak{g})$ *is a subgroup of* $\text{TAut}_n(\mathfrak{g})$.

*Proof.* Let $(a_1, .., a_n)$ and $(b_1, ..., b_n)$ belong to $\text{TAut}_n^{an}(\mathfrak{g})$. Let $\alpha(r), \beta(r) \in \mathbb{R}_+\{\{r\}\}_0$ be germs such that the identities $|a_i(x_1, ..., x_n)| \preceq \alpha(\sup_i |x_i|)$, $|b_i(x_1, ..., x_n)| \preceq \beta(\sup_i |x_i|)$ hold. Then

$$\begin{aligned}|c_i(x_1, ..., x_n)| &\preceq f_M(|a_i(x_1, ..., x_n)| + |b_i(e^{\text{ad}\, a_1}(x_1), ..., e^{\text{ad}\, a_n}(x_n))|) \\ &\preceq f_M(\alpha(\sup_i |x_i|) + \beta(e^{M\alpha(\sup_i |x_i|)} \sup_i |x_i|)) = \gamma(\sup_i |x_i|),\end{aligned}$$

where $f_M(u) = \frac{1}{M} f(Mu)$ and $\gamma(r) = f_M(\alpha(r) + e^{M\alpha(r)}\beta(r))$ has nonzero radius of convergence. Here we use the compatibility of norms with composition: namely, if $f \in E[[V_1 \times .. \times V_n]]_0$ and $g_i \in V_i[[W]]_0$, with $|f(v_1, ..., v_n)| \preceq \alpha(|v_1|, ..., |v_n|)$ and $|g_i(w)| \preceq \beta_i(|w|)$, then $h := f \in (g_1, ..., g_n) \in E[[W]]_0$ and $|h(w)| \preceq \alpha \circ (\beta_1, ..., \beta_n)(|w|)$. We also use the non-decreasing properties of elements of $\mathbb{R}_+[[r_1,...,r_n]]_0$ (i.e., if $F \in \mathbb{R}_+[[u_1,...,u_k]]_0$ and $u_i, u'_i \in \mathbb{R}_+[[r_1,...,r_l]]_0$ with $u_i \preceq u'_i$, then $F(u_1,...) \preceq F(u'_1,...)$. So $(a_1,...,a_n)(b_1,...,b_n) \in \mathrm{TAut}_n^{an}(\mathfrak{g})$.

If now $(a_1,...,a_n) \in \mathrm{TAut}_n^{an}(\mathfrak{g})$, then its inverse $(b_1,...,b_n)$ in $\mathrm{TAut}_n(\mathfrak{g})$ is uniquely determined by the identities

$$b_i(x_1,...,x_n) = -a_i(e^{\mathrm{ad}\, b_1(x_1,...,x_n)}(x_1),....,e^{\mathrm{ad}\, b_n(x_1,...,x_n)}(x_n)).$$

Let us show that each $b_i(x_1,...,x_n)$ is an analytic germ. For this, we define inductively the sequence $b^{(k)} = (b_1^{(k)},...,b_n^{(k)})$ by $b^{(0)} = (0,..,0)$, and

$$b_i^{(k+1)}(x_1,...,x_n) = -a_i(e^{\mathrm{ad}\, b_1^{(k)}(x_1,...,x_n)}(x_1),....,e^{\mathrm{ad}\, b_n^{(k)}(x_1,...,x_n)}(x_n)).$$

One checks that $b^{(k)} = b^{(k-1)} + O(x^k)$, so the sequence $(b^{(k)})_{k\geq 0}$ converges in the formal series topology; the limit $b$ is then the inverse of $a = (a_1,...,a_n)$.

Let us now set $\beta_k := \sup_i |b_i^{(k)}|$ (if $u_i(r) = \sum_{k\geq 0} u_{i,k}r^k \in \mathbb{R}_+[[r]]$ is a finite family, we set $(\sup_i u_i)(r) := \sum_{k\geq 0}(\sup_i u_{i,k})r^k$). We then have

$$|b_i^{(k+1)}(x_1,...,x_n)| \preceq \alpha(\sup_i |e^{\mathrm{ad}\, b_i^{(k)}(x_1,...,x_n)}(x_i)|) \preceq \alpha(e^{M\beta_k(\sup_i |x_i|)} \sup_i |x_i|),$$

so $\beta_{k+1}(r) \preceq \alpha(e^{\beta_k(r)}r)$.

We now define a sequence $(\gamma_k)_{k\geq 0}$ of elements of $\mathbb{R}_+[[r]]_0$ by $\gamma_0 = 0$,

$$\gamma_{k+1}(r) = \alpha(e^{M\gamma_k(r)}r).$$

As the exponential function, mutiplication by $r$ and $\alpha$ are non-decreasing, we have $\beta_k \preceq \gamma_k$. On the other hand, we have $\gamma_k(r) = \gamma_{k-1}(r) + O(r^k)$, so the sequence $(\gamma_k)_k$ converges in $\mathbb{R}_+[[r]]_0$ (one also checks that this sequence is non-decreasing). Its limit $\gamma$ then satisfies

$$\gamma(r) = \alpha(e^{M\gamma(r)}r). \tag{23}$$

It is easy to show that (23) determines $\gamma(r) \in \mathbb{R}[[r]]_0$ uniquely. On the other hand, the function $(\gamma, r) \mapsto \gamma - \alpha(e^{M\gamma}r) =: F(\gamma, r)$ is analytic at the neighborhood of $(0,0)$, with differential at this point $\partial_\gamma F(0,0)d\gamma + \partial_r F(0,0)dr = d\gamma - M\alpha'(0)dr$. We may then apply the implicit function theorem and use the fact that the $d\gamma$-component of $dF(0,0)$ is nonzero to derive the existence of an analytic function $\gamma_{an}(r)$ satisfying (23). By the uniqueness of solutions of (23), we get that the expansion of $\gamma_{an}$ is $\gamma$, so $\gamma \in \mathbb{R}_+\{\{r\}\}_0$.

Now $|b_i^{(k)}(x_1,...,x_n)| \preceq \beta_k(\sup_i |x_i|) \preceq \gamma_k(\sup_i |x_i|) \preceq \gamma(\sup_i |x_i|)$, so by taking the limit $k \to \infty$, $|b_i(x_1,...,x_k)| \preceq \gamma(\sup_i |x_i|)$, which implies that $b_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0$, as wanted. □

According to [AT], we have a bijection

$$\kappa : \mathrm{TAut}_n \to \mathfrak{tder}_n, \quad g \mapsto \ell - g\ell g^{-1},$$

where $\ell$ is the derivation given by $x_i \mapsto x_i$.

Set $\mathfrak{tder}_n(\mathfrak{g}) := \{(u_1,...,u_n)|u_i(x_1,...,x_n) \in \mathfrak{g}[[\mathfrak{g}^n]]_0\}$, and $\mathfrak{tder}_n^{an}(\mathfrak{g}) := \{(u_1,...,u_n)|u_i \in \mathfrak{g}\{\{\mathfrak{g}^n\}\}_0\} \subset \mathfrak{tder}_n(\mathfrak{g})$. We have maps $\mathrm{TAut}_n \to \mathrm{TAut}_n(\mathfrak{g})$, $\mathfrak{tder}_n \to \mathfrak{tder}_n(\mathfrak{g})$ induced by the specialization of formal series.

**Lemma 9.3.** *1) There exists a map $\kappa_{\mathfrak{g}} : \mathrm{TAut}_n(\mathfrak{g}) \to \mathfrak{tder}_n(\mathfrak{g})$, such that the diagram*

$$\begin{array}{ccc} \mathrm{TAut}_n & \stackrel{\kappa}{\to} & \mathfrak{tder}_n \\ \downarrow & & \downarrow \\ \mathrm{TAut}_n(\mathfrak{g}) & \stackrel{\kappa_{\mathfrak{g}}}{\to} & \mathfrak{tder}_n(\mathfrak{g}) \end{array}$$

*commutes.*

*2) This map restricts to a map $\kappa_{\mathfrak{g}}^{an} : \mathrm{TAut}_n^{an}(\mathfrak{g}) \to \mathfrak{tder}_n^{an}(\mathfrak{g})$.*

*Proof.* 1) If $a_i, b_i \in \hat{\mathfrak{f}}_n$ are such that $g = [\![e^{b_1}, ..., e^{b_n}]\!]$, $g^{-1} = [\![e^{a_1}, ..., e^{a_n}]\!]$, then $\kappa(g) = u = [\![u_1, ..., u_n]\!]$, with

$$u_i(x_1, ..., x_n) = (\frac{1 - e^{\mathrm{ad}\, a_i}}{\mathrm{ad}\, a_i}(\dot{a}_i))(e^{\mathrm{ad}\, b_1(x_1,...,x_n)}(x_1), ..., e^{\mathrm{ad}\, b_n(x_1,...,x_n)}(x_n))$$

and $\dot{a}_i = \ell(a_i) = \frac{d}{dt}_{|t=1} a_i(tx_1, ..., tx_n)$. So we define $\kappa_{\mathfrak{g}}$ by the same formula, where $\dot{a}_i$ is now defined as $\frac{d}{dt}_{|t=1} a_i(tx_1, ..., tx_n)$ (or $\sum_{k\geq 0} k a_i^k$, where $a_i^k$ is the degree $n$ part of $a_i$).

2) If the functions $a_i, b_i$ are analytic germs, then so is $\dot{a}_i$ and therefore also each $u_i$. □

Recall also from [AT] that if $\mu \in \mathrm{TAut}_2$, $\mu(x*y) = x+y$ and $J(\mu) = \langle r(x)+r(y)-r(x+y)\rangle$ (i.e., $\mu \in \mathrm{SolKV}(\mathbf{k})$), then $u := -\kappa(\mu^{-1}) = [\![A(x, y), B(x, y)]\!]$ satisfies:
(KV1) $x + y - y * x = (1 - e^{-\mathrm{ad}\, x})(A(x, y)) + (e^{\mathrm{ad}\, y} - 1)(B(x, y))$,
(KV3) $j(u) = \langle \phi(x) + \phi(y) - \phi(x * y)\rangle$, where $\phi(t) = tr'(t)$.

Let $\Phi_{\mathrm{KZ}}$ be the KZ associator, $\tilde{\Phi}_{\mathrm{KZ}}(a, b) := \Phi_{\mathrm{KZ}}(a/(2\pi\, \mathrm{i}), b/(2\pi\, \mathrm{i})) \in M_1(\mathbb{C})$ and $\mu_{\mathrm{KZ}} := \mu_{\tilde{\Phi}_{\mathrm{KZ}}}$. Let $u_{\mathrm{KZ}} := \kappa(\mu_{\mathrm{KZ}}^{-1})$. Then $J(\mu_{\mathrm{KZ}}) = \langle r_{\mathrm{KZ}}(x) + r_{\mathrm{KZ}}(y) - r_{\mathrm{KZ}}(x * y)\rangle$, where $r_{\mathrm{KZ}}(u) = -\sum_{n\geq 2}(2\pi\, \mathrm{i})^{-n}\zeta(n)u^n/n$, therefore

$$j(u_{\mathrm{KZ}}) = \langle \phi_{\mathrm{KZ}}(x) + \phi_{\mathrm{KZ}}(y) - \phi_{\mathrm{KZ}}(x * y)\rangle,$$

where $\phi_{\mathrm{KZ}}(u) = -\sum_{n\geq 2}(2\pi\, \mathrm{i})^{-n}\zeta(n)u^n$. Now the real part of this function (obtained by taking the real part of the coefficients of $u^n$) is

$$\phi^{\mathbb{R}}_{\mathrm{KZ}}(u) = \frac{1}{2}(\frac{u}{e^u - 1} - 1 + \frac{u}{2}).$$

Let us now set $u_{\mathbb{R}} := [\![A_{\mathbb{R}}(x, y), B_{\mathbb{R}}(x, y)]\!]$, where the real part is taken with respect to the natural real structure on $\mathfrak{f}_2^{\mathbb{C}}$. Then by the linearity of (KV1), (KV3), we have:

$$\text{(KV1)} \quad x + y - y * x = (1 - e^{-\mathrm{ad}\, x})(A_{\mathbb{R}}(x, y)) + (e^{\mathrm{ad}\, y} - 1)(B_{\mathbb{R}}(x, y))$$

$$\text{(KV3)} \quad j(u_{\mathbb{R}}) = \frac{1}{2}\langle \frac{x}{e^x - 1} + \frac{y}{e^y - 1} - \frac{x * y}{e^{x*y} - 1} - 1\rangle.$$

9.3. **Analytic aspects to the KV conjecture (proof of Theorem 2.8).** Recall that $\log \tilde{\Phi}_{\mathrm{KZ}} \in \hat{\mathfrak{f}}_2$. We denote the specialization of this series to the Lie algebra $\mathfrak{g}$ as $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}[[\mathfrak{g}^2]]_0$.

**Proposition 9.4.** $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}}$ *is an analytic germ, i.e.,* $(\log \tilde{\Phi}_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}\{\{\mathfrak{g}^2\}\}_0$.

*Proof.* Recall that $A_2 = U(\mathfrak{f}_2)$ is the free associative algebra in $a, b$. For $x \in A_2$, set

$$|x|_{A_2} := \sup_{N\geq 1} \sup_{m_1,m_2\in M_N(\mathbb{C})} ||x(m_1, m_2)||.$$

Here $||.||$ is an algebra norm on $M_N(\mathbb{C})$. Then $|x|_{A_2}$ is $\leq \sum_{I\in\sqcup_{n\geq 0}\{0,1\}^n} |x_I|$, where $x = \sum_I x_I e_I$, and for $I = (i_1, ..., i_n)$, $e_I = e_{i_1}...e_{i_n}$, $e_0 = a$, $e_1 = b$. It follows from the Amitsur–Levitsky theorem ([AL]) that $(|x|_{A_2} = 0) \Rightarrow (x = 0)$; indeed, by this theorem, $x(m_1, m_2) = 0$ for $m_1, m_2 \in M_N(\mathbb{C})$ implies: (a) that $x$ is in the 2-sided ideal generated by $ab - ba$ if $N = 1$; (b) that $x = 0$ if $N > 1$. It follows that $|.|_{A_2}$ is an algebra norm[9] on $A_2$, in particular $|xy|_{A_2} \leq |x|_{A_2}|y|_{A_2}$.

We then define a vector space norm $|.|_{\mathfrak{f}_2}$ on $\mathfrak{f}_2$ by $|x|_{\mathfrak{f}_2} := |x|_{A_2}$; we have $|[x, y]|_{\mathfrak{f}_2} \leq 2|x|_{\mathfrak{f}_2}|y|_{\mathfrak{f}_2}$.

For $\mathbf{n} = (n_1, ..., n_d) \in \mathbb{N}^d$, and $f$ a function on $(\mathfrak{f}_2)^d$ (resp., $\mathbb{R}^d$), we denote by $f(\xi_1, ..., \xi_d)_{\mathbf{n}}$ (resp., $f(t_1, ..., t_d)_{\mathbf{n}}$) the $\mathbf{n}$-multihomogeneous part of $f$, which we view as a multihomogeneous polynomial on $(\mathfrak{f}_2)^d$ (resp., $\mathbb{R}^d$).

**Lemma 9.5.** *For any* $\mathbf{n}$*, we have the identity*

$$|\log(e^{\xi_1}...e^{\xi_d})_{\mathbf{n}}|_{\mathfrak{f}_2} \leq ((\log(2 - e^{t_1+...+t_d})^{-1})_{\mathbf{n}})_{t_1=|\xi_1|_{\mathfrak{f}_2},...,t_d=|\xi_d|_{\mathfrak{f}_2}}.$$

[9] We will not use $(|x|_{A_2} = 0) \Rightarrow (x = 0)$, so our proof is independent of the Amitsur–Levitsky theorem.

*Proof of Lemma.* We have for any $\mathbf{n}$, $|\xi_1^{n_1}...\xi_d^{n_d}|_{A_2} \leq |\xi_1|_{\mathfrak{f}_2}^{n_1}...|\xi_d|_{\mathfrak{f}_2}^{n_d}$ so

$$|(e^{\xi_1}...e^{\xi_d}-1)_{\mathbf{n}}|_{A_2} \leq ((e^{t_1+...+t_d}-1)_{\mathbf{n}})_{t_1=|\xi_1|_{\mathfrak{f}_2},...,t_d=|\xi_d|_{\mathfrak{f}_2}}.$$

Then $\log(e^{\xi_1}...e^{\xi_d})_{\mathbf{n}} = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}\sum_{(\mathbf{n}_1,...,\mathbf{n}_k)|\mathbf{n}_1+...+\mathbf{n}_k=\mathbf{n}}(e^{\xi_1}...e^{\xi_d}-1)_{\mathbf{n}_1}...(e^{\xi_1}...e^{\xi_d}-1)_{\mathbf{n}_k}$ so

$$|\log(e^{\xi_1}...e^{\xi_d})_{\mathbf{n}}|_{A_2} \leq \Big(\sum_{k\geq 1}\frac{1}{k}\sum_{\mathbf{n}_1+...+\mathbf{n}_k=\mathbf{n}}(e^{t_1+...+t_d}-1)_{\mathbf{n}_1}...(e^{t_1+...+t_d}-1)_{\mathbf{n}_d}\Big)_{t_1=|\xi_1|_{\mathfrak{f}_2},...,t_d=|\xi_d|_{\mathfrak{f}_2}}$$

$$= \Big(\sum_{k\geq 1}\frac{1}{k}((e^{t_1+...+t_d}-1)^k)_{\mathbf{n}}\Big)_{t_1=|\xi_1|_{\mathfrak{f}_2},...,t_d=|\xi_d|_{\mathfrak{f}_2}} = ((\log(2-e^{t_1+...+t_d})^{-1})_{\mathbf{n}})_{t_1=|\xi_1|_{\mathfrak{f}_2},...,t_d=|\xi_d|_{\mathfrak{f}_2}}.$$

□

Let $a(t)$ be an function $[0,1]\to\hat{\mathfrak{f}}_2$ of the form $a(t)=\sum_{k\geq 1}a_k(t)$, where $a_k(t)\in\mathfrak{f}_2[k]$ (here $k$ is the total degree in $a,b$) and $\int_0^1|a_k(t)|_{\mathfrak{f}_2}dt<\infty$. Let $u_0,u_1$ be solutions of $u'(t)=a(t)u(t)$ with $u_0(0)=u_1(1)=1$, and $U:=u_1^{-1}u_0$.

**Lemma 9.6.** *For $n\geq 1$, let $(\log U)_n$ the degree $n$ (in $a,b$) part of $\log U$. Then*

$$\sum_{n\geq 1}|(\log U)_n|_{\mathfrak{f}_2}r^n \preceq \log(2-e^{\sum_{k\geq 1}r^k\int_0^1|a_k(t)|_{\mathfrak{f}_2}dt})^{-1}.$$

*Proof of Lemma.* Let Lie$(n)$ be the multilinear part of $\mathfrak{f}_n$ in the generators $x_1,...,x_n$. We denote by $w_n(x_1,...,x_n)\in$ Lie$(n)$ the multilinear part of $\log(e^{x_1}...e^{x_n})$.

Let now $\alpha_n$ be the coefficient of $t_1...t_n$ in the expansion of $\log(2-e^{t_1+...+t_n})^{-1}$ (this is also the $n$th derivative at $t=0$ of $\log(2-e^t)^{-1}$). Specializing Lemma 9.5 for $\mathbf{n}=(1,..,1)$, we get the identity

$$|w_n(\xi_1,...,\xi_n)|_{\mathfrak{f}_2}\leq\alpha_n|\xi_1|_{\mathfrak{f}_2}...|\xi_n|_{\mathfrak{f}_2}$$

for $\xi_1,...,\xi_n\in\mathfrak{f}_2$.

Now $\log U$ expands as

$$\log U=\sum_{n\geq 0}\int_{0<t_1<...<t_n<1}w_n(a(t_1),...,a(t_n))dt_1...dt_n$$

(see e.g. [EG]). It follows that

$$(\log U)_k=\sum_{n\geq 0}\sum_{k_1,...,k_n|\sum_i k_i=k}\int_{0<t_1<...<t_n<1}w_n(a_{k_1}(t_1),...,a_{k_n}(t_n))dt_1...dt_n$$

and therefore

$$|(\log U)_k|_{\mathfrak{f}_2}\leq\sum_{n\geq 0}\alpha_n\sum_{k_1,...,k_n|\sum_i k_i=k}\int_{0<t_1<...<t_n<1}|a_{k_1}(t_1)|_{\mathfrak{f}_2}...|a_{k_n}(t_n)|_{\mathfrak{f}_n}dt_1...dt_n.$$

Now the generating series for the r.h.s. is $\log(2-e^{\sum_{k\geq 1}r^k\int_0^1|a_k(t)|_{\mathfrak{f}_2}dt})^{-1}$, proving the result.

□

According to [Dr], Section 2, if we set

$$a(t):=\sum_{k\geq 0,l\geq 1}\frac{1}{k!l!(2\pi\mathrm{i})^{k+l+1}}\frac{(-\log(1-t))^k(-\log t)^l}{t-1}(\mathrm{ad}\,b)^k(\mathrm{ad}\,a)^l(b),$$

then $\tilde\Phi_{KZ}=U$. We have $|(\mathrm{ad}\,b)^k(\mathrm{ad}\,a)^l(b)|_{\mathfrak{f}_2}\leq k+l+2\leq 2^{k+l+1}$, so

$$|a_n(t)|\leq\sum_{k\geq 0,l\geq 1,k+l+1=n}\frac{1}{\pi^{k+l+1}k!l!}\frac{(-\log(1-t))^k(-\log t)^l}{1-t}$$

Then we have the inequality of formal series in $r$

$$\sum_{n\geq 1} r^n \int_0^1 |a_n(t)|_{\mathfrak{f}_2} dt \preceq \int_0^1 \sum_{k\geq 0, l\geq 1} \frac{r^{k+l+1}}{\pi^{k+l+1} k! l!} \frac{(-\log(1-t))^k (-\log t)^l}{1-t} dt$$
$$= \frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt.$$

Now the identity $\int_0^1 t^a (1-t)^b dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$, valid for $\Re(a), \Re(b) > -1$, implies that if $\Re(r) < 0$, then

$$\frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt = \frac{1}{2} \big(1 - \frac{\Gamma(1-2r)^2}{\Gamma(1-4r)}\big).$$

This implies that the radius of convergence of $\frac{r}{\pi} \int_0^1 (1-t)^{-1-\frac{r}{\pi}} (t^{-\frac{r}{\pi}} - 1) dt$ is $1/4$, so this series belongs to $\mathbb{R}_+\{\{r\}\}_0$. Plugging this in Lemma 9.6, we get

$$\sum_{n\geq 0} |(\log \tilde\Phi_{\mathrm{KZ}})_n|_{\mathfrak{f}_2} r^n \preceq \log(2 - e^{\frac{1}{2}\big(1 - \frac{\Gamma(1-2r)^2}{\Gamma(1-4r)}\big)})^{-1},$$

where the series in the r.h.s. lies in $\mathbb{R}_+\{\{r\}\}_0$ (being a composition of two series in $\mathbb{R}_+\{\{r\}\}_0$).

Let us now prove that $(\log \tilde\Phi_{\mathrm{KZ}})^{\mathfrak{g}} \in \mathfrak{g}\{\{\mathfrak{g}^2\}\}_0$ is an analytic germ. By Ado's theorem, there exists a injective morphism $\rho : \mathfrak{g} \to M_N(\mathbf{k})$, where $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, hence an injective morphism $\tilde\rho : \mathfrak{g} \to M_N(\mathbb{C})$. Equip $\mathfrak{g}$ with the norm $|x|_{\mathfrak{g}} := ||\tilde\rho(x)||$. We recall that all the norms on $\mathfrak{g}$ are equivalent, so it will suffice to prove analyticity w.r.t. $|.|_{\mathfrak{g}}$.

The degree $n$ part of the series $(\log \tilde\Phi_{\mathrm{KZ}})^{\mathfrak{g}}$ is the specialization to $\mathfrak{g}$ of $(\log \tilde\Phi_{\mathrm{KZ}})_n$. Now if $\psi \in \mathfrak{f}_2[n]$ and $\psi^{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is its specialization to $\mathfrak{g}$, we have $|\psi^{\mathfrak{g}}(x,y)|_{\mathfrak{g}} = ||\psi(\tilde\rho(x), \tilde\rho(y))|| \leq |\psi|_{\mathfrak{f}_2} \sup(||\tilde\rho(x)||, ||\tilde\rho(y)||)^n = |\psi|_{\mathfrak{f}_2} \sup(|x|_{\mathfrak{g}}, |y|_{\mathfrak{g}})^n$, therefore $|\psi^{\mathfrak{g}}| \leq |\psi|_{\mathfrak{f}_2}$. We then have

$$\sum_{n\geq 0} |(\log \tilde\Phi_{\mathrm{KZ}})^{\mathfrak{g}}_n| r^n \preceq \sum_{n\geq 0} |(\log \tilde\Phi_{\mathrm{KZ}})_n|_{\mathfrak{f}_2} r^n \preceq \log(2 - e^{\frac{1}{2}\big(1 - \frac{\Gamma(1-2r)^2}{\Gamma(1-4r)}\big)})^{-1};$$

together with the fact that the series in the right has positive radius of convergence, this implies the analyticity of the series $(\log \tilde\Phi_{\mathrm{KZ}})^{\mathfrak{g}}$. □

Proposition 9.4, together with the local analyticity of the CBH series, implies that the specialization of $\mu_{\tilde\Phi_{\mathrm{KZ}}}$ belongs to $\mathrm{TAut}_2^{an}(\mathfrak{g})$. It follows that $A(x,y)$, $B(x,y)$ are analytic germs, and so

(KV2) $(A^{\mathbb{R}}, B^{\mathbb{R}})$ is an analytic germ $\mathfrak{g}^2 \to \mathfrak{g}^2$.

All this implies that $(A^{\mathbb{R}}, B^{\mathbb{R}})$ is a solution of the 'original' KV conjecture (as formulated in [KV]) and proves 1) in Theorem 2.8.

Let us now prove Theorem 2.8, 2). One checks easily that if $(A,B)$ is a solution of the 'original' KV conjecture, then $(A_s, B_s) := (A + s(\log(e^x e^y) - x), B + s(\log(e^x e^y) - y))$ is a family of solutions. In fact, if $\mu \in \mathrm{SolKV}(\mathbf{k})$ and $[\![A,B]\!] = -\kappa(\mu^{-1})$, then $[\![A_s, B_s]\!] = -\kappa(\mu_{-s}^{-1})$, where $\mu_s := \mathrm{Inn}(e^{s(x+y)}) \circ \mu$; this corresponds to the action of 'trivial', degree 1 element of $\mathfrak{krv}$ on SolKV (see[AT]).

Finally, let us prove Theorem 2.8, 3). Let $\sigma$ be the antilinear automorphism of $\hat{\mathfrak{f}}_2$ such that $\sigma(x) = -y$, $\sigma(y) = -x$. The series $\Phi_{\mathrm{KZ}}(a,b)$ is real, therefore $\overline{\tilde\Phi_{\mathrm{KZ}}(a,b)} = \tilde\Phi_{\mathrm{KZ}}(-a,-b)$ (the bar denotes the complex conjugation). This implies that $\mu_{\mathrm{KZ}} \circ \sigma = \mathrm{Inn}(e^{-(x+y)/2}) \circ \sigma \circ \mu_{\mathrm{KZ}}$. Using $\sigma \circ \ell \circ \sigma^{-1} = \ell$ and $\ell(x+y) = x+y$, we get

$$(\mu_{\mathrm{KZ}} \circ \sigma \circ \mu_{\mathrm{KZ}}^{-1}) \circ \ell \circ (\mu_{\mathrm{KZ}} \circ \sigma \circ \mu_{\mathrm{KZ}}^{-1})^{-1} = \ell + \mathrm{inn}(\frac{1}{2}(x+y)),$$

where $\mathrm{inn}(x+y)$ is the inner derivation $z \mapsto [x+y, z]$ of $\hat{\mathfrak{f}}_2$. Using now $\mu_{\mathrm{KZ}}^{-1}(x+y) = \log(e^x e^y)$, we get

$$(\sigma \circ \mu_{\mathrm{KZ}}^{-1}) \circ \ell \circ (\sigma \circ \mu_{\mathrm{KZ}}^{-1})^{-1} = \mu_{\mathrm{KZ}}^{-1} \circ \ell \circ \mu_{\mathrm{KZ}} + \mathrm{inn}(\frac{1}{2}\log(e^x e^y)).$$

Since $\sigma\circ\ell\circ\sigma^{-1}=\ell$, $\mu^{-1}\circ\ell\circ\mu-\ell=-[\![A_{\mathrm{KZ}},B_{\mathrm{KZ}}]\!]$ and $\mathrm{inn}(\frac12\log(e^xe^y))=[\![\frac12(\log(e^xe^y)-x),\frac12(\log(e^xe^y)-y)]\!]$

$$\sigma\circ[\![A_{\mathrm{KZ}},B_{\mathrm{KZ}}]\!]\circ\sigma^{-1}=[\![A_{\mathrm{KZ}},B_{\mathrm{KZ}}]\!]-[\![\frac12(\log(e^xe^y)-x),\frac12(\log(e^xe^y)-y)]\!].$$

This implies

$$(B_{\mathrm{KZ}}(-y,-x),A_{\mathrm{KZ}}(-y,-x))=(A_{\mathrm{KZ}}(x,y),B_{\mathrm{KZ}}(x,y))-(\frac12(\log(e^xe^y)-x),\frac12(\log(e^xe^y)-y)).$$

If now $(A',B'):=(A_{\mathrm{KZ}},B_{\mathrm{KZ}})-\frac14(\log(e^xe^y)-x,\log(e^xe^y)-y)$, this implies

$$(B'(-y,-x),A'(-y,-x))=(A'(x,y),B'(x,y)),$$

which by taking real parts implies $(B_{-1/4}(-y,-x),A_{-1/4}(-y,-x))=(A_{-1/4}(x,y),B_{-1/4}(x,y))$, proving Theorem 2.8, 3).

## Appendix A. Results on centralizers

### A.1. The centralizer of $t_{ij}$ in $\mathfrak{t}_n$.

**Proposition A.1.** *Let $i<j\in[n]$. If $x\in\mathfrak{t}_n$ is such that $[x,t_{ij}]=0$, then there exists $\lambda\in\mathbf{k}$ and $y\in\mathfrak{t}_{n-1}$ such that $x=\lambda t_{ij}+y^{ij,1,2,\ldots,\check{i},\ldots,\check{j},\ldots,n}$.*

*Proof.* We may and will assume that $i=1,j=2$. We then prove the result by induction on $n$. It is obvious when $n=2$. Assume that it has been proved at step $n-1$ and let us prove it at step $n$. We have $\mathfrak{t}_n=\mathfrak{t}_{n-1}\oplus\mathfrak{f}_{n-1}$, where $\mathfrak{t}_{n-1}$ is the Lie subalgebra generated by the $t_{ij}$, $i\neq j\in\{1,\ldots,n-1\}$ and $\mathfrak{f}_{n-1}$ is freely generated by the $t_{1n},\ldots,t_{n-1,n}$. Both $\mathfrak{t}_{n-1}$ and $\mathfrak{f}_{n-1}$ are Lie subalgebras of $\mathfrak{t}_n$, stable under the inner derivation $[t_{12},-]$. Then if $x\in\mathfrak{t}_n$ is such that $[t_{12},x]=0$, we decompose $x=x'+f$, with $x'\in\mathfrak{t}_{n-1}$, $f\in\mathfrak{f}_{n-1}$, $[t_{12},x']=[t_{12},f]=0$. By the induction hypothesis, we have $x'=\lambda t_{12}+(y')^{12,3,\ldots,n-1}$, where $y'\in\mathfrak{t}_{n-2}$ and $\lambda\in\mathbf{k}$.

Let us set $x_i=t_{in}$ for $i=1,\ldots,n-1$. The derivation $[t_{12},-]$ of $\mathfrak{f}_{n-1}$ is given by $x_1\mapsto[x_1,x_2]$, $x_2\mapsto[x_2,x_1]$, $x_i\mapsto 0$ for $i>2$. In terms of generators $y_1=x_1$, $y_2=x_1+x_2$, $y_3=x_3\ldots$, $y_{n-1}=x_{n-1}$, it is given by $y_1\mapsto[y_1,y_2]$, $y_i\mapsto 0$ for $i>1$.

**Lemma A.2.** *The kernel of the derivation $y_1\mapsto[y_1,y_2]$, $y_i\mapsto 0$ for $i>1$ of $\mathfrak{f}_{n-1}$ coincides with the Lie subalgebra $\mathfrak{f}_{n-2}\subset\mathfrak{f}_{n-1}$ generated by $y_2,\ldots,y_{n-1}$.*

*Proof of Lemma.* Let us prove that the kernel of the induced derivation of $U(\mathfrak{f}_{n-1})$ is $U(\mathfrak{f}_{n-2})$. We have a linear isomorphism $U(\mathfrak{f}_{n-1})\simeq\oplus_{k\geq1}U(\mathfrak{f}_{n-2})^{\otimes k}$, whose inverse takes $u_1\otimes\ldots\otimes u_k$ to $u_1y_1u_2y_1\ldots y_1u_k$. The derivation $[t_{12},-]$ of $U(\mathfrak{f}_{n-1})$ is then transported to the direct sum of the endomorphisms of $U(\mathfrak{f}_{n-2})^{\otimes k}$

$$u\mapsto(y_2^{(2)}+\ldots+y_2^{(k)})u-u(y_2^{(1)}+\ldots+y_2^{(k-1)}) \tag{24}$$

(this is 0 of $k=1$; $y_2^{(i)}=1^{\otimes i-1}\otimes y_2\otimes 1^{\otimes k-i}$; we make use of the algebra structure of $U(\mathfrak{f}_{n-2})^{\otimes k}$). Each of these endomorphisms has degree 1 for the filtration of $U(\mathfrak{f}_{n-2})^{\otimes k}$ induced by the PBW filtration of $U(\mathfrak{f}_{n-2})$ (the part of degree $\leq d$ of $U(\mathfrak{f}_{n-2})$ for this filtration consists of combinations of products of $\leq d$ elements of $\mathfrak{f}_{n-2}$) and the associated graded endomorphism of $S(\mathfrak{f}_{n-2})^{\otimes k}$ is the multiplication by $y_2^{(k)}-y_2^{(1)}$, which is injective if $k\geq1$, so (24) is injective for $k\geq1$; the kernel of the direct sum of maps (24) therefore coincides with the degree 1 part $U(\mathfrak{f}_{n-2})$, which transports to $U(\mathfrak{f}_{n-2})\subset U(\mathfrak{f}_{n-1})$. So the kernel of the derivation $[t_{12},-]$ of $U(\mathfrak{f}_{n-1})$ is $U(\mathfrak{f}_{n-2})$. The kernel of the derivation $[t_{12},-]$ of $\mathfrak{f}_{n-1}$ is then $\mathfrak{f}_{n-1}\cap U(\mathfrak{f}_{n-2})=\mathfrak{f}_{n-2}$. □

*End of proof of Proposition A.1.* It follows that $f$ expresses as $P(t_{1n}+t_{2n},t_{3n},\ldots,t_{n-1,n})$. Then if we set $f':=P(t_{1,n-1},\ldots,t_{n-2,n-1})$, we get $f=(f')^{12,3,\ldots,n}$ so $x=x'+f=\lambda t_{12}+((y')^{1,2,\ldots,n-1}+f')^{12,3,\ldots,n}$, as wanted. □

A.2. **The centralizer of $x_{ij}$ in $\mathrm{PB}_n$.**

**Proposition A.3.** *If $g \in \mathrm{PB}_n(\mathbf{k})$ commutes with $x_{12}$, then there exists $\lambda \in \mathbf{k}$ and $h \in \mathrm{PB}_{n-1}(\mathbf{k})$ such that $g = x_{12}^{\lambda} h^{\widetilde{12},3,\ldots,n}$.*

Since $x_{ij}$ is conjugated to $x_{12}$, it is easy to derive from this the centralizer of $x_{ij}$ in $\mathrm{PB}_n(\mathbf{k})$.

*Proof.* Note that $x_{12}$ commutes with the image of $\mathrm{PB}_{n-1}(\mathbf{k}) \to \mathrm{PB}_n(\mathbf{k})$, $h \mapsto h^{\widetilde{12},3,\ldots,n}$, so that $U_0 := \{x_{12}^{\lambda} h^{\widetilde{12},3,\ldots,n} | h \in \mathrm{PB}_{n-1}(\mathbf{k}), \lambda \in \mathbf{k}\}$ is an algebraic subgroup of $\mathrm{PB}_n(\mathbf{k})$. Let $U \subset \mathrm{PB}_n(\mathbf{k})$ be the centralizer of $x_{12}$; then $U_0 \subset U$, and we need to prove that $U_0 = U$.

We have $U_0 = \exp(\mathfrak{u}_0)$, $U = \exp(\mathfrak{u})$, where $\mathfrak{u}_0 = \mathbf{k} \log x_{12} \oplus \mathrm{Im}(\mathfrak{pb}_{n-1} \stackrel{\widetilde{12},3,\ldots,n}{\to} \mathfrak{pb}_n)$ and $\mathfrak{u} = \{x \in \mathfrak{pb}_n | [\log x_{12}, x] = 0\}$, where $\mathfrak{pb}_n := \mathrm{Lie}\,\mathrm{PB}_n(\mathbf{k})$. Then the lower central series defines a complete decreasing filtration of $\mathfrak{pb}_n$, with $F^1\mathfrak{pb}_n = \mathfrak{pb}_n$ and $F^{i+1}\mathfrak{pb}_n = [\mathfrak{pb}_n, F^i\mathfrak{pb}_n]$. The associated graded Lie algebra is $\mathfrak{t}_n$, i.e., $\mathfrak{t}_n = \oplus_{i\geq 1} \mathfrak{t}_n[i] = \oplus_{i \geq 1} F^i\mathfrak{pb}_n / F^{i+1}\mathfrak{pb}_n$.

Set $F^i\mathfrak{u} := \mathfrak{u} \cap F^i\mathfrak{pb}_n$, $F^i\mathfrak{u}_0 := \mathfrak{u}_0 \cap F^i\mathfrak{pb}_n$. We will prove that the images of $F^i\mathfrak{u}_0$ and $F^i\mathfrak{u}$ in $\mathfrak{t}_n[i]$ coincide. Clearly, $\mathrm{Im}(F^i\mathfrak{u}_0 \to \mathfrak{t}_n[i]) \subset \mathrm{Im}(F^i\mathfrak{u} \to \mathfrak{t}_n[i])$.

Conversely, projecting the identity $[\log x_{12}, x] = 0$ modulo $F^{i+1}\mathfrak{pb}_n$, we get

$$\mathrm{Im}(F^i\mathfrak{u} \to \mathfrak{t}_n[i]) \subset \{x \in \mathfrak{t}_n[i] | [t_{12}, x] = 0\}, \tag{25}$$

and since $x \mapsto x^{\widetilde{12},\ldots,n}$ takes $F^i\mathfrak{pb}_{n-1}$ to $F^i\mathfrak{pb}_n$, we have $(F^i\mathfrak{pb}_{n-1})^{\widetilde{12},\ldots,n} \subset F^i\mathfrak{u}_0$ if $i > 1$ and $(F^1\mathfrak{pb}_{n-1})^{\widetilde{12},\ldots,n} \oplus \mathbf{k}\log x_{12} \subset F^1\mathfrak{u}_0$; projecting these inclusions, modulo $F^{i+1}\mathfrak{pb}_n$, we get

$$\mathrm{Im}(F^i\mathfrak{u}_0 \to \mathfrak{t}_n[i]) \supset \mathfrak{t}_{n-1}[i]^{12,\ldots,n} \text{ if } i > 1 \text{ and } \mathrm{Im}(F^1\mathfrak{u}_0 \to \mathfrak{t}_n[i]) \supset \mathfrak{t}_{n-1}[1]^{12,\ldots,n} \oplus \mathbf{k}t_{12}. \tag{26}$$

Using (25), (26) and Proposition A.1, we get $\mathrm{Im}(F^i\mathfrak{u} \to \mathfrak{t}_n[i]) \subset \mathrm{Im}(F^i\mathfrak{u}_0 \to \mathfrak{t}_n[i])$. It follows that these spaces are equal, which implies (as both $\mathfrak{u}_0$ and $\mathfrak{u}$ are closed for the topology of $\mathfrak{pb}_n$) that $\mathfrak{u}_0 = \mathfrak{u}$. So $U_0 = U$. □

**Remark A.4.** One can also prove Proposition A.3 similarly to Proposition A.1, by induction on $n$ and using the fact that the automorphism $\mathrm{Ad}\, x_{12}$ of the topologically free group generated by the $x_{in}$ identifies with the automorphism $\exp(\mathrm{ad}\, t_{12})$ of the topologically free Lie algebra generated by the $t_{in}$ (using the identification $(x_{1n}, x_{1n}x_{2n}, x_{3n}, ..., x_{n-1,n}) \leftrightarrow (e^{t_{1n}}, e^{t_{1n}+t_{2n}}, e^{t_{3n}}, ..., e^{t_{n-1,n}})$).

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